Uncollapsing the wavefunction by undoing quantum measurements

Andrew N. Jordan[1] and Alexander N. Korotkov[2]

[1]Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627, USA

[2] Department of Electrical Engineering, University of California, Riverside, CA 92521, USA

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We review and expand on recent advances in theory and experiments concerning the problem of wavefunction uncollapse: Given an unknown state that has been disturbed by a generalized measurement, restore the state to its initial configuration. We describe how this is probabilistically possible with a subsequent measurement that involves erasing the information extracted about the state in the first measurement. The general theory of abstract measurements is discussed, focusing on quantum information aspects of the problem, in addition to investigating a variety of specific physical situations and explicit measurement strategies. Several systems are considered in detail: the quantum double dot charge qubit measured by a quantum point contact (with and without Hamiltonian dynamics), the superconducting phase qubit monitored by a SQUID detector, and an arbitrary number of entangled charge qubits. Furthermore, uncollapse strategies for the quantum dot electron spin qubit, and the optical polarization qubit are also reviewed. For each of these systems the physics of the continuous measurement process, the strategy required to ideally uncollapse the wavefunction, as well as the statistical features associated with the measurement is discussed. We also summarize the recent experimental realization of two of these systems, the phase qubit and the polarization qubit.

I. INTRODUCTION

The irreversibility of quantum measurement is an axiomatic property of textbook quantum mechanics.[1] In his famous article *Law without law*, John Wheeler expresses the idea with poetic flare: “We are dealing with [a quantum] event that makes itself known by an irreversible act of amplification, by an indelible record, an act of registration.”[2] However, it has been gradually recognized that the textbook treatment of an instantaneous wavefunction collapse is really a very special case of what is in general a dynamical process - continuous quantum measurement. Continuous measurements do not project the system immediately into an eigenstate of the observable, but describe a process whereby the collapse happens over a period of time.[3] The fact that continuous measurement is a dynamical process with projective measurement as a special case, leads us to ask whether the irreversibility of quantum measurement is also a special case. The purpose of this paper is to review and expand on recent developments in this area of research, showing that it is possible to undo a quantum measurement, thereby uncollapsing the wavefunction, and to describe this physics in detail for both the abstract and concrete physical realizations.

This paper follows our earlier work on the subject,[4,5] as well as other papers investigating similar questions.[6,7] Wavefunction uncollapse teaches us several things about the fundamentals of quantum mechanics. First, there is a notion that wavefunction represents many possibilities, but that reality is created by measurement. The fact that the effects of measurement can be undone suggests that this idea is flawed, or at least too simplistic. If you create reality with quantum measurements, does undoing them erase the reality you created? Secondly, there is a wide-spread belief that quantum measurement is nothing more than a decoherence process. This suggests that the superposition never really collapses; it only appears to collapse. What actually happens, according to this idea, is that all the information about the system disperses into the environment: when a quantum system interacts with a classical measuring device, it becomes irreversibly entangled with all the particles that make up the measuring device and its surroundings. The uncollapse of the wavefunction demonstrates that decoherence theory cannot be the whole story, because a true decoherence process is irreversible.[8] The perspective we take in this paper further advances the ‘quantum Bayesian’ point of view, where the quantum state is nothing more than a reflection of our information about the system. When we receive more information about the system, the state changes or collapses not because of any mysterious forces, but simply as a result of Bayesian updating.

Our work is indirectly related to the ‘quantum eraser’ of Scully and Drühl.[9] There, the which-path information of a particle is encoded in the quantum state of an atom, resulting in a destruction of interference fringe visibility. On the other hand, if the which-path information of the particle is erased, the interference fringes are restored. Both the Scully proposal and our proposal erase information. However, there is an important difference. In order for the uncollapsing procedure to work, we have to erase the information that was already extracted classically. In the ‘quantum eraser’, only potentially extractable information is erased.

While the first part of this paper deals with the abstract idea of uncollapse, and formalizes its properties in terms of generalized quantum measurements, a great deal of the paper deals with actual measurement processes in specific physical contexts. This brings to mind the saying of Asher Peres: “Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory”.[10] Following Peres’ dictum, we discuss measurement pro-

cesses in a variety of solid state systems, where there has been remarkable experimental progress in recent years. Quantum coherence has been demonstrated to occur in a controllable fashion in systems such as semiconductor quantum dots and superconducting Josephson junctions. We will discuss the physics of measurement in these systems, as well as concrete strategies for uncollapsing the wavefunction. It should be stressed that two of these proposals (a superconducting phase qubit and optical polarization qubit) have now been implemented in the laboratory, providing conclusive demonstration of wavefunction uncollapse.[11]

The paper is organized as follows. In Sec. II we describe in detail what we mean by the undoing of a quantum measurement (Sec. II A), and give a general treatment of the physics, using the formalism of positive operator-valued measures (POVMs). This is done both for pure states (Sec. II B) and mixed states (Sec. II C). In Sec. II D we discuss the interpretation of wavefunction uncollapse, and what it tells us about quantum information. Sec. III begins discussion of the physical implementation of this physics, with a treatment of the double quantum dot qubit, monitored by a quantum point contact. The measurement dynamics is discussed in III A and we discuss the uncollapsing strategy and statistical predictions in Sec. III B for the case of a qubit undergoing measurement dynamics only. The results are discussed in III C and compared with the abstract results. In Sec. III D we discuss the statistical features of the time required to wait until the wavefunction is uncollapsed. We generalize to the case of the finite-Hamiltonian qubit undergoing the uncollapse process in Sec. III E, and derive results for the success probability in that case. We switch to a new physical system in Sec. IV, the superconducting phase qubit. In Sec. IV A we discuss how the measurement process works for the phase qubit, and in Sec. IV B we discuss the uncollapsing strategy, corresponding success probability, and the recent experimental realization, which demonstrated the uncollapsing. In Sec. V, we generalize to the case of many charge qubits, and discuss an explicit procedure to undo any generalized measurement. We discuss recent developments in the theory and experiments of measurement reversal in Sec. VI and conclude in Sec. VII.

II. GENERAL THEORY OF UNCOLLAPSING

A. Preliminary discussion

Our goal is to restore an initial quantum state disturbed by measurement. However, it is important to discuss what exactly we mean by that. For example, if we start with a *known* pure state $|\psi_{in}\rangle$ and perform a textbook projective measurement, then it is trivial to restore the initial state: since we also know the post-measurement wavefunction $|\psi_m\rangle$, we just need to apply a unitary operation which transfers $|\psi_m\rangle$ into $|\psi_{in}\rangle$. If we start with a known mixed state, then its restoration after a projective measurement is a little more involved;[12] however such a procedure still can be easily analyzed using standard quantum mechanics and classical probability theory.

In this paper we consider a different, non-trivial situation: we assume that an arbitrary initial state is *unknown to us*, and we still want to restore it after the measurement disturbance. To make this idea more precise, we consider a contest between the uncollapse *proponent* Plato, and an uncollapse *skeptic* Socrates. Socrates prepares a quantum system in any state he likes, but it is unknown to Plato. Socrates sends the state to Plato, who makes some measurement on the system, verified by the *arbiter* Aristotle. Plato then tries to undo the measurement. If Plato judges that the attempt succeeded, the system is returned to Socrates, with the claim that it is in the original state. Socrates is then allowed to try and find a contradiction in any way he likes, with the whole process monitored by Aristotle. If a contradiction can be found, then he can claim to refute the uncollapse claim, but in the absence of contradiction, the uncollapse claim stands. If Socrates would like to try again to find a contradiction, or if Plato judges that his undoing attempt was unsuccessful (and does not return a state), then Socrates prepares a new (still unknown to Plato) state, and the competition continues. If Socrates cannot find a contradiction after many rounds of the competition, then Plato will win the contest, and will have successfully demonstrated the uncollapsing of the quantum state.

A slightly different but equivalent situation is when we know the initial state, but our uncollapsing procedure must be independent of the initial state (so we can pretend that it is unknown), and therefore the uncollapsing should restore *any* initial state in the same way. This formulation is most appropriate for a real experiment demonstrating the uncollapsing. Finally, we may consider the more general case where the measured system is entangled with another system, and we wish to restore the initial state of the compound system without any access to its second part.

The traditional statement of irreversibility of a quantum measurement can be traced to the fact that it may be described as a mathematical projection. Projection is a many-to-one mapping in the Hilbert space, and therefore the same post-measurement state generally corresponds to (infinitely) many initial states.[13] It is therefore impossible to undo a projective measurement.

However, the situation is different for a general[14] (POVM-type) measurement, which typically corresponds to a one-to-one mapping $|\psi_{in}\rangle \to |\psi_m\rangle$ in the Hilbert space of wavefunctions (in this paper we consider only "ideal" measurements which do not introduce extra decoherence). In this case the post-measurement wavefunction $|\psi_m\rangle$ can still be associated with the unique initial state $|\psi_{in}\rangle$, and a well-defined inverse mapping exists mathematically. This makes the uncollapsing possible

in principle. Since the inverse mapping is typically nonunitary, it cannot be realized as an evolution with a suitable Hamiltonian. However, it can be realized using another POVM-type measurement with a specific ("lucky") result.

B. Formalism for wavefunctions

Let us first consider a pure initial state $|\psi_{in}\rangle$, and postpone a generalization to mixed states until Sec. II C. In the formalism of a general (ideal) quantum measurement[14] which transfers pure states into pure states, the measurement with result m is associated with the linear Kraus operator M_m, so that the probability of result m is

$$P_m(|\psi_{in}\rangle) = ||M_m|\psi_{in}\rangle\,||^2, \tag{1}$$

where $||\dots||$ denotes the norm of the state, and the (conditioned) state after measurement is

$$|\psi_m\rangle = \frac{M_m|\psi_{in}\rangle}{\sqrt{P_m(|\psi_{in}\rangle)}}, \tag{2}$$

where the denominator makes $|\psi_m\rangle$ properly normalized. (Very often people prefer to omit this denominator and work with non-normalized states; this makes the mapping linear.) The operators $E_m = M_m^\dagger M_m$ (called POVM elements[14]) are Hermitian and positive semidefinite by construction; these operators must obey the completeness relation $\sum_m E_m = \mathbf{1}$, which ensures that the total probability of all measurement results is unity. A measurement operator M_m can always be written as

$$M_m = U_m\sqrt{E_m}, \tag{3}$$

where U_m is a unitary operator (an important special case is when $M_m = \sqrt{E_m}$; this corresponds to the "quantum Bayes theorem"[15]).

Now let us discuss wavefunction uncollapse in this general and abstract context. The state disturbance rule (2) is typically a nonunitary one-to-one map in the Hilbert space. To undo the measurement with known result m, we have to realize a physical process corresponding to the nonunitary inverse operator M_m^{-1}, multiplied by an arbitrary constant (which is not important because of the normalization). This can be accomplished with another measurement, possibly together with unitary operations. As shown below, we can realize measurement undoing if the second measurement realizes a Krauss operator of the form

$$L = CU_L E_m^{-1/2} V_L, \tag{4}$$

where U_L and V_L are any unitary operators, and C is an unspecified constant that will be discussed later. (The operator L also has a decomposition of the form $U\sqrt{E}$, but with a different POVM element E.) The uncollapse then consists of three steps: first, the unitary operator $V_L^\dagger U_m^\dagger$ is applied to reverse the unitary part of M_m and prepare for the second measurement. Next the measurement operator L is applied. Finally the unitary operator $U_L^\dagger$ is applied to reverse the remaining unitary part of L. We can now see the effect of the uncollapsing operation on the state $|\psi_m\rangle$ by applying Eq. (2) and the unitaries to find

$$|\psi_f\rangle = \frac{U_L^\dagger L V_L^\dagger U_m^\dagger|\psi_m\rangle}{||U_L^\dagger L V_L^\dagger U_m^\dagger|\psi_m\rangle||} = |\psi_{in}\rangle, \tag{5}$$

thus restoring the original state, because $U_L^\dagger L V_L^\dagger U_m^\dagger M_m = C$, which is removed by the normalization. (The phase of C is not important, since it affects only the overall phase of the wavefunction.)

However, in order for the operator L to be physically realizable, the operator $L^\dagger L$ must belong to another complete set of POVM elements, and therefore all its eigenvalues must not exceed unity (otherwise some states will be assigned probabilities that are above unity; notice that the eigenvalues are non-negative automatically). Since $L^\dagger L = |C|^2 V_L^\dagger E_m^{-1} V_L$, its eigenvalues are directly related to the eigenvalues $p_i^{(m)}$ of the operator E_m. Expressing $E_m = \sum_i p_i^{(m)}|i\rangle\langle i|$, where the eigenvectors $|i\rangle$ form an orthonormal basis, the eigenvectors of $L^\dagger L$ are obviously $V_L^\dagger|i\rangle$, and the corresponding eigenvalues are $|C|^2/p_i^{(m)}$. Since all these eigenvalues must not exceed 1, we find the following inequality on $|C|^2$,

$$|C|^2 \le \min_i p_i^{(m)} = \min P_m, \tag{6}$$

where $\min P_m$ is the probability of the result m, minimized over all possible states $|\psi_{in}\rangle$ in the Hilbert space. The equality of $\min P_m$ to $\min_i p_i^{(m)}$ follows from Eq. (1).

There is no guarantee that the uncollapse can be accomplished deterministically, since we rely on a measurement with a specific result, corresponding to the operator L. We can calculate the uncollapse success probability P_S from Eq. (1), with $M_m \to L$ and $|\psi_{in}\rangle \to V_L^\dagger U_m^\dagger|\psi_m\rangle$,

$$P_S = ||LV_L^\dagger U_m^\dagger|\psi_m\rangle||^2 = \left|\left|\frac{C|\psi_{in}\rangle}{\sqrt{P_m(|\psi_{in}\rangle)}}\right|\right|^2 = \frac{|C|^2}{P_m(|\psi_{in}\rangle)}. \tag{7}$$

Now using the bound (6) for $|C|^2$, we find the bound for the success probability of uncollapsing after the first measurement with result m:

$$P_S \le \frac{\min P_m}{P_m(|\psi_{in}\rangle)}, \tag{8}$$

where the denominator is the probability of the result m for a given initial state, while the numerator is this probability minimized over all possible initial states.

The bound (8) is one of the most important results (notice a similar result in Ref. [6]) and deserves a discussion. First, this bound is exact in the sense that it is

achievable by the optimal uncollapsing procedure. This is because the uncollapsing operator with $|C| = \sqrt{\min P_m}$ is still a physically allowed operator. As we will see later, the upper bound (8) is achievable in real experimental setups. However, non-optimal uncollapsing procedures, especially involving a sequence of measurements, can lead to smaller success probabilities (an example of non-optimal uncollapsing has been discussed in Ref. [16]). An analysis of the procedures with an arbitrary sequence of measurements and unitary operations is similar to the above: the corresponding measurement and unitary operators should simply be multiplied.

Notice that the success probability (7) and the inequality (8) depends on the initial state, which is unknown to the person performing the uncollapsing (Plato, see description in the previous subsection). Therefore, the success probability P_S can be calculated by the man who knows what the initial state is (Socrates), while Plato can only estimate P_S; for example, he can calculate the worst-case scenario (the minimum of P_S over the accessible Hilbert space) or can calculate the average of P_S over all possible initial states (this procedure will be discussed in the next subsection).

Recalling the fact that it is not possible to undo a fully collapsed state due to the nature of projective measurement, the uncollapsing probability P_S should decrease with increasing strength of the first measurement. Qualitatively, a stronger measurement is one that tends to a projection, as the uncertainty in the measurement decreases. Mathematically, this means that some eigenvalues of E_m become closer to 0. As a consequence, $\min P_m$ becomes smaller [see Eq. (6)], therefore lowering the upper bound for P_S. For a projective measurement $P_S = \min P_m = 0$, thus making the state uncollapse impossible.

It is interesting to discuss the case when the initial state $|\psi_{in}\rangle$ is known to belong to a certain subspace of the Hilbert space, and we therefore wish to restore states only in this subspace. In this case, the calculation of $\min P_m$ should be limited to this subspace, which may increase the bound (8) for the success probability P_S. A trivial example of such a situation is when the initial state $|\psi_{in}\rangle$ is known to Plato. Here it is not necessary to minimize P_m over all possible initial states in Eq. (8), because the set of possible states consists of only one (known) state, thus allowing uncollapsing with 100% probability. This is exactly the case discussed at the beginning of Sec. II A.

C. Formalism for density matrices

So far we have dealt only with pure states; however, it is very simple to generalize the uncollapsing formalism to include density matrices. In this case the initial density matrix ρ_{in} is transformed by the first measurement into the state[14]

$$\rho_m = \frac{M_m \rho_{in} M_m^\dagger}{P_m}, \tag{9}$$

where the probability P_m of the measurement result m is

$$P_m(\rho_{in}) = \mathrm{Tr}(M_m^\dagger M_m \rho_{in}). \tag{10}$$

Using the uncollapsing procedure previously discussed and using the same measurement operator L given by Eq. (4), we find that the uncollapsed state

$$\rho_f = \frac{U_L^\dagger L V_L^\dagger U_m^\dagger \rho_m U_m V_L^\dagger L^\dagger U_L}{\mathrm{Tr}(L^\dagger L V_L U_m^\dagger \rho_m U_m V_L^\dagger)} = \rho_{in} \tag{11}$$

coincides with the initial state. The uncollapsing success probability P_S is equal to the denominator in Eq. (11), and satisfies the relation

$$P_S = |C|^2 / P_m(\rho_{in}) \tag{12}$$

[as in Eq. (7)]. The constant $|C|^2$ is still limited by the inequality (6), and therefore the probability of success has the upper bound

$$P_S \leq \frac{\min P_m}{P_m(\rho_{in})}, \tag{13}$$

which is the same as the bound (8), except for the new notation in the denominator, which reminds us of the possibly mixed initial state. The minimization of P_m in the numerator should now be performed over the space of all possible initial mixed states; however, the result obviously coincides with the minimization over the pure states only. Similar to the case discussed in Sec. II B, the inequality (13) is the exact bound; it is achieved by the optimal uncollapsing procedure, which maximizes $|C|$.

If the initial state is pure, then the formalism of this subsection is trivially equivalent to the formalism of Sec. II B. It becomes more general in the case when the "actual" initial state is mixed; for example, this happens when the initial state has been in contact with an unmonitored environment or Socrates prepares a state by a blind random choice from a set of pure states. A more interesting case for the result (13) is when the measured system is entangled with another system, which does not evolve by itself. Then the formalism can be applied to the compound system; however, the measurement probability P_m depends only on the reduced initial density matrix, traced over the entangled second part. Therefore, in the entangled bipartite case the uncollapsing procedure restores the state of the whole system, while the success probability P_S is given by Eq. (13) with ρ_{in} being the reduced density matrix.

Another advantage of Eq. (13) in comparison with Eq. (8) is the following. In the derivation of both results the initial state is the "actual" initial state, which is known to Socrates, but typically unknown to Plato. However, as we will prove below, Eq. (13) can still be used by Plato in a somewhat different sense: with ρ_{in} being understood as an *averaged* density matrix representing a distribution of possible initial states. In this

case, Eq. (13) gives the uncollapsing probability averaged over this distribution. For example, if Plato knew that Socrates' strategy is to prepare one of 2 possible (nonorthogonal) states $|\psi_1\rangle, |\psi_2\rangle$, with probabilities $\mathcal{P}$ and $1-\mathcal{P}$, then he could find the average undoing probability in two ways. The first method is that he could simply average the undoing probabilities of the two states (also taking into account the information acquired in the first measurement, see below). Alternatively, he could recall that the random state preparation described above is equivalent to considering the initial density matrix $\rho_{in} = \mathcal{P}|\psi_1\rangle\langle\psi_1| + (1-\mathcal{P})|\psi_2\rangle\langle\psi_2|$, and then apply[17] the result (13) to this density matrix. In this way, in the absence of any information, Plato could estimate his typical success rate by calculating (13) for a fully mixed state, invoking the principle of indifference.[18]

In the general case the above statement, that both ways of computing the averaged undoing probability are equivalent, can be proven both logically and explicitly. For the logical proof we notice that Plato's judgment of successful undoing does not depend on whether or not Socrates knows the randomly picked state; therefore, the average probability of the cases judged to be successful should be the same in both situations (whether or not Socrates knows what the state is). Now let us also prove this statement explicitly, thus checking that our formalism is self-consistent. Suppose the initial state is prepared by Socrates by choosing randomly from a set of initial states $\rho^{(k)}$ with probabilities $\mathcal{P}_k$ (the most natural case is when initial states are pure, $\rho^{(k)} = |\psi_k\rangle\langle\psi_k|$; however, this is not necessary). Then the bound for the average probability of uncollapsing success $P_S^{(\mathrm{av})}$ is the average of the bounds (13):

$$P_S^{(\mathrm{av})} \leq \sum\nolimits_k \frac{\min P_m}{P_m(\rho^{(k)})}\,\mathcal{P}'_k. \tag{14}$$

Notice, however, that $\mathcal{P}'_k$ is the posterior probability distribution given the result m, which is different from $\mathcal{P}_k$. We may now invoke the classical Bayes rule[18,19]

$$\mathcal{P}(k|m) = \frac{\mathcal{P}(m|k)\mathcal{P}_k}{\sum_{\tilde{k}} \mathcal{P}(m|\tilde{k})\mathcal{P}_{\tilde{k}}} \tag{15}$$

to relate the posterior $\mathcal{P}'_k = \mathcal{P}(k|m)$ to the prior $\mathcal{P}_k$ and the conditional probability $\mathcal{P}(m|k) = P_m(\rho^{(k)})$ to have result m given state k, so that

$$\mathcal{P}'_k = \frac{P_m(\rho^{(k)})\mathcal{P}_k}{\sum_{\tilde{k}} P_m(\rho^{(\tilde{k})})\mathcal{P}_{\tilde{k}}}. \tag{16}$$

Substituting Eq. (16) into Eq. (14) and using $\sum_k \mathcal{P}_k = 1$ in the numerator, we obtain

$$P_S^{(\mathrm{av})} \leq \frac{\min P_m}{\sum_k P_m(\rho^{(k)})\mathcal{P}_k} = \frac{\min P_m}{P_m(\rho^{(\mathrm{av})})}, \tag{17}$$

where $\rho^{(\mathrm{av})} = \sum_k \rho^{(k)}\mathcal{P}_k$ is the averaged initial state. This ends the proof that Eq. (13) can be used for an unknown initial state, with ρ_{in} being understood as the average of all possible initial states.

D. Uncollapsing probability, information, and an irreversibility measure

We defined the success probability P_S as a probability to uncollapse the post-measurement state ρ_m. We now wish to start counting the overall success probability $\tilde{P}_S$ from the time before the first measurement, so that $\tilde{P}_S$ is the probability of the pair of events: measurement with result m and then successful uncollapsing. Using Eqs. (12) and (13) we easily find the relation

$$\tilde{P}_S = P_m(\rho_{in})P_S = |C|^2 \tag{18}$$

and the upper bound

$$\tilde{P}_S \leq \min P_m. \tag{19}$$

Notice that $\tilde{P}_S$ is independent of the initial state. While this property seems to be somewhat surprising, we will see later why it is rather obvious.

If we now wish to consider all possible results of the first measurement, and perform different uncollapsing procedures for each measurement result, then the total probability of uncollapsing $\tilde{P}_S^{\mathrm{total}}$ is bounded as

$$\tilde{P}_S^{\mathrm{total}} \leq \sum\nolimits_m \min P_m, \tag{20}$$

and is also independent of the initial state.

The bounds (19) and (20) are exact and reachable by optimal uncollapsing procedures. The bound (20) indicates that $1 - \sum_m \min P_m$ can be used as a *measure of irreversibility* (collapse strength) due to the measurement operation.

Now let us discuss the relationship between the uncollapsing procedure and our knowledge of the initial state. While uncollapsing is possible even if we know nothing about the initial state of the system, at first glance it seems like we gain some knowledge about the initial state in the process. This leads to the following interesting paradox, initially considered by Royer.[20] By doing both a measurement and unmeasurement, one can seemingly learn something about the initial state without disturbing it. Then by repeating the measurement + unmeasurement process many times, even though the probability of such an event rapidly decreases to zero, the successful event would lead to essentially perfect knowledge of the initial state, leaving the state itself perfectly intact! One could then violate a host of known results, such as the no-cloning theorem. The resolution of the paradox lies in the fact that the pair of measurement and unmeasurement actually brings exactly *zero information*. Uncollapsing the state can only occur when the information in the second measurement exactly contradicts the information gained in the first measurement, thus nullifying it. This can happen in weak quantum measurements because there is uncertainty about the system in the measurement result. It is to the extent that this ambiguity exists that it is possible to undo the weak measurement. Let us examine this in more detail.

We learn something about a pre-measurement state when the measurement result depends on the state. The measurement with result m brings some information about ρ_{in} because the probability $P_m(\rho_{in})$ depends on the initial state ρ_{in}. The ability to successfully uncollapse the state *also* brings some information about the initial state because the uncollapsing probability $P_S = |C|^2/P_m(\rho_{in})$ also depends on ρ_{in}. However, the collapse-uncollapse probability $\tilde{P}_S$ of observing *both* the result m followed by a successful uncollapse is independent of ρ_{in} – see Eq. (18). Therefore, the combined effect of partial collapse and uncollapse brings no information about the initial state.

More quantitatively, we can use the same framework as at the end of Sec. II C in order to track the information gain during the procedure. Suppose Plato assigns an initial distribution $\mathcal{P}_k$ of possible initial states as a statistical prior, to be updated as more information comes in. The measurement with result m brings in this information, so Plato updates his prior to the posterior distribution $\mathcal{P}'_k$ [see Eq. (16)]. Calculating in a similar way the distribution $\mathcal{P}''_k$ after the pair of the measurement and unmeasurement results, we find

$$\mathcal{P}''_k = \frac{P_S(\rho^{(k)})\mathcal{P}'_k}{\sum_{\tilde{k}} P_S(\rho^{(\tilde{k})})\mathcal{P}'_{\tilde{k}}} = \frac{\tilde{P}_S(\rho^{(k)})\mathcal{P}_k}{\sum_{\tilde{k}} \tilde{P}_S(\rho^{(k)})\mathcal{P}_{\tilde{k}}}, \quad (21)$$

where $P_S(\rho)$ and $\tilde{P}_S(\rho)$ denote respectively the probabilities of uncollapsing (13) and a combined collapse-uncollapse pair (18) for the initial state ρ. However, as we have already stressed, $\tilde{P}_S(\rho)$ is independent of the initial state ρ, and therefore cancels out of the expression (21). This fact (and the normalization of the prior $\{P_k\}$) restores the initial prior distribution, $\mathcal{P}''_k = \mathcal{P}_k$, and therefore Plato has learned nothing, thus avoiding the paradox. Reversing the logic, in order to avoid the paradox, $\tilde{P}_S$ must be independent of the initial state, as found in Eqs. (18) and (19).

III. DOUBLE-QUANTUM-DOT CHARGE QUBIT

Consider Fig. 1 illustrating a double-quantum-dot (DQD) qubit, measured continuously by a symmetric quantum point contact (QPC). This setup has been extensively studied in earlier papers, both theoretically[21] and experimentally.[22] The measurement is characterized by the average currents I_1 and I_2 corresponding to the qubit state $|1\rangle$ and $|2\rangle$ (the double-dot electron being in one dot, or the other), and by the shot noise spectral density S_I.[23] We treat the additive detector shot noise as a Gaussian, white, stochastic process, and assume the detector is in the weakly responding regime, $|\Delta I| \ll I_0$, where $\Delta I = I_1 - I_2$ and $I_0 = (I_1 + I_2)/2$, with QPC voltage much larger than all other energy scales, so that the measurement process can be described by the quantum Bayesian formalism.[24]

A. Measurement dynamics for a non-evolving qubit

We begin for simplicity with the assumption that there is no qubit Hamiltonian evolution, so that the qubit state evolves due to the measurement only (this can also be effectively done using "kicked" quantum nondemolition (QND) measurements[26]). As was shown in Ref. 24, at low temperature the QPC is an ideal quantum detector (which does not decohere the measured qubit), so that the evolution of the qubit density matrix ρ due to continuous measurement preserves the "murity"[27] $\mathcal{M}$ while the diagonal matrix elements evolve according to the classical Bayes rule.[19] We define the electrical current through the QPC averaged in a time t as $\bar{I}(t) = [\int_0^t I(t')\, dt']/t$, and the quantum Bayesian equations read

$$\rho_{11}(t) = \frac{\rho_{11}(0)P_1(\bar{I})}{\rho_{11}(0)P_1(\bar{I}) + \rho_{22}(0)P_2(\bar{I})}, \quad (22)$$

$$\rho_{22}(t) = \frac{\rho_{22}(0)P_2(\bar{I})}{\rho_{11}(0)P_1(\bar{I}) + \rho_{22}(0)P_2(\bar{I})}, \quad (23)$$

$$\mathcal{M} = \rho_{12}/\sqrt{\rho_{11}\rho_{22}} = \text{const}, \quad (24)$$

where the conditional (Gaussian) probability densities of a current $\bar{I}$ realization, given that the qubit is in $|1\rangle, |2\rangle$ are

$$P_{1,2}(\bar{I}) = \sqrt{t/\pi S_I}\, \exp[-(\bar{I} - I_{1,2})^2 t/S_I]. \quad (25)$$

Equations (22) and (23) may be simplified by noting

$$\frac{\rho_{11}(t)}{\rho_{22}(t)} = \frac{\rho_{11}(0)}{\rho_{22}(0)}\, e^{2r(t)}, \quad (26)$$

where we define the dimensionless *measurement result* as

$$r(t) = \frac{t\Delta I}{S_I}\,[\bar{I}(t) - I_0] = \frac{\Delta I}{S_I}\int_0^t [I(t') - I_0]\, dt'. \quad (27)$$

Notice that $r(t)$ is closely related to the total charge passed through the QPC, and therefore $r(t)$ accumulates in time. For times much longer than the "measurement time" $T_M = 2S_I/(\Delta I)^2$ (the time scale required to obtain a signal-to-noise ratio of 1), the average current $\bar{I}$ tends to either I_1 or I_2 because the probability density $P(\bar{I})$ of a particular $\bar{I}$ is

$$P(\bar{I}) = \sum_{i=1,2} \rho_{ii}(0)P_i(\bar{I}). \quad (28)$$

Therefore $r(t)$ tends to $\pm\infty$, continuously collapsing the state to either $|1\rangle$ (for $r \to \infty$) or $|2\rangle$ (for $r \to -\infty$). Importantly, for the special case when the initial state is pure, the state remains pure during the entire process.

This set of DQD measurement dynamics can be seen to be related to the general measurement formalism[14] described in the previous section in the following way.[27] For a fixed time t the measurement result m can be associated with the averaged current $\bar{I}$ (or, equivalently, with the dimensionless quantity r). The Kraus operator M_m

is then diagonal in the measurement basis $|1\rangle$ and $|2\rangle$ and has matrix elements $\sqrt{P_{1,2}(\bar{I})}$. Notice that P_m now describes the probability density of the result $\bar{I}$ instead of probability, because the measurement result becomes a continuous variable.

B. Uncollapsing for the charge qubit

In order to describe how to uncollapse the charge qubit state, we note that if $r(t) = 0$ at some moment t, then the qubit state becomes exactly the same as it was initially, $\rho(t) = \rho(0)$, as follows from Eqs. (26) and (24). This of course must be the case if $t = 0$, *i.e.* before the measurement began, but is equally valid for some later time. To see why this is so from the informational point of view, we note that in the absence of noise, the measurement result from states $|1\rangle, |2\rangle$ would simply be $r_{1,2}(t) = \pm t/T_M$. With the noise present, the measurement outcome $r(t) = 0$ splits in half the difference between states $|1\rangle$ and $|2\rangle$. Such an outcome corresponds to an equal statistical likelihood of the states $|1\rangle$ and $|2\rangle$, and therefore provides no information about the state of the qubit.

Suppose the outcome of a measurement is r_0, partially collapsing the qubit state toward either state $|1\rangle$ (if $r_0 > 0$), or state $|2\rangle$ (if $r_0 < 0$). The previous "no information" observation suggests the following strategy for uncollapsing: continue measuring, with the hope that after some time t the stochastic result of the second measurement $r_u(t)$ becomes equal to $-r_0$, so the total result $r(t) = r_0 + r_u(t)$ is zero, and therefore the initial qubit state is fully restored. If this happens, the measuring device is immediately switched off and the uncollapsing procedure is successful (Fig. 1). However $r(t)$ may never cross the origin, and then the uncollapsing attempt fails.

This strategy requires the observation of a particular measurement result that may never materialize. The strategy shifts the randomness to the amount of time that needs to elapse in order to find the desired measurement result. Of course, in a given realization the measurement result could take on the desired value multiple times, so we will take as our strategy to turn off the detector the first time the measurement result takes on $r = 0$. In the classical stochastic physics this is known as a first passage process,[28] the theory of which is well developed and will be used below.

In order to analyze the uncollapsing strategy performance, in particular to find the success probability P_S, it is important to notice that the off-diagonal elements of the qubit density matrix ρ do not come into play when we consider the detector output $I(t)$ (this is true only in the case of zero or QND-eliminated qubit Hamiltonian; we will shortly generalize to the finite qubit Hamiltonian case). As a result, the quantum problem can be exactly reduced to a classical problem by substituting $\rho_{11}(t)$ and $\rho_{22}(t)$ with classical probabilities, evolving in the course of measurement according to the classical Bayes rule,

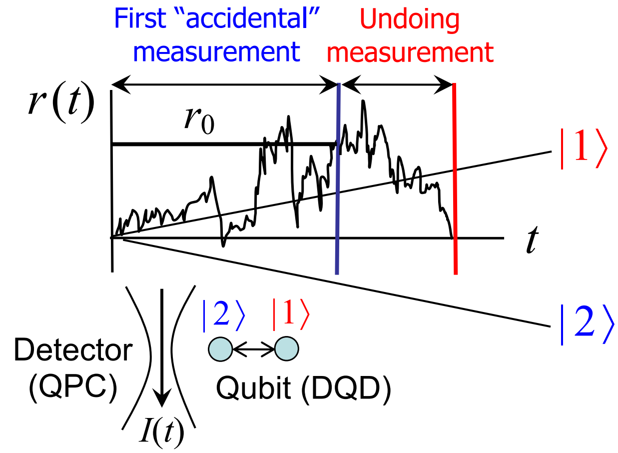

FIG. 1: (Color online). Illustration of the uncollapsing procedure for the charge qubit. The slanted lines indicate the deterministic output of the detector in the absence of noise, if the qubit is in state $|1\rangle$ or $|2\rangle$. The initial measurement yields the result r_0. The detector is again turned on, hoping that at some future time the measurement result $r(t) = r_0 + r_u(t)$ crosses the origin, at which time the detector is turned off, successfully erasing the information obtained in the first measurement, and restoring the initial qubit state.

while evolution of the off-diagonal elements $\rho_{12} = \rho_{21}^*$ can be found automatically from the murity conservation law (24). We therefore model the qubit by a classical bit with probability $p_1 = \rho_{11}(0)$ of being prepared in state "1" and probability $p_2 = \rho_{22}(0)$ of being in state "2". If the bit is in state "1", the dimensionless measurement result $r(t)$ evolves as a random walk with diffusion coefficient $D = (\Delta I)^2/4S_I = 1/2T_M$ and drift velocity $v_1 = (\Delta I)^2/2S_I = 1/T_M$ [see Eqs. (25) and (27)]. For the bit state "2" the random walk of $r(t)$ has the same diffusion coefficient but the opposite drift velocity $v_2 = -(\Delta I)^2/2S_I$. We are given the fact that the first part of the measurement had the result r_0 (i.e. we select only such realizations). We need to analyze the stochastic behavior of the total measurement result $r(t)$ during the second part of measurement, with most attention to the crossing of the zero line $r(t) = 0$ [for convenience we shift $t = 0$ to the beginning of the second measurement, so that $r(0) = r_0$].

Let us find the probability P_S of such a crossing. We will first obtain it in a simple way, and then reproduce the result in a more complicated way, which will also allow us to analyze the statistics of the waiting time. For definiteness take $r_0 > 0$ (this will be extended later). Then the result $r(t)$ will necessarily cross 0 if the bit is actually in the state "2", because in this case $r(t = \infty) = -\infty$ while $r(t = 0) = r_0 > 0$. The probability of being in the state "2" is $\tilde{p}_2 = p_2 e^{-r_0}/(p_1 e^{r_0} + p_2 e^{-r_0})$ from (26), which differs from p_2 because of the Bayesian update. If the bit is in state "1" (this happens with probability $\tilde{p}_1 = 1 - \tilde{p}_2$), then $r(t = \infty) = +\infty$ and the crossing of 0 may never happen; however, it is still possible with some

probability P_C, which depends on r_0, and also on D and v_1. To find P_C, let us consider an infinitesimal time step dt and model the diffusion by discrete jumps in r of magnitude $\Delta r = \pm\sqrt{2D\,dt}$. After a step dt, the coordinate will then shift to one of two positions, $r = r_\pm$, where $r_\pm = r_0 + v_1 dt \pm \sqrt{2D\,dt}$. Each of these new coordinates will have its own probability of eventually crossing the origin, $P_C(r_\pm)$. Because the diffusive dynamics is generated by choosing either r_+ or r_- with equal weighting, it follows in the limit $dt \to 0$ that

$$P_C(r_0) = \sum_\pm \frac{1}{2}\, P_C(r_\pm). \tag{29}$$

Expanding $P_C(r_\pm)$ in this relation in a Taylor series, we find from the linear in dt term that

$$D\,\partial^2_{r_0} P_C = -v_1\,\partial_{r_0} P_C, \tag{30}$$

where ∂_{r_0} and $\partial^2_{r_0}$ denote the first and second derivatives with respect to r_0. Taking into account that $P_C = 1$ for $r_0 = 0$ and $P_C = 0$ for $r_0 = \infty$, the above differential equation may be easily solved to find $P_C = \exp(-v_1 r_0/D) = \exp(-2r_0)$. Now collecting the probabilities of the zero line crossing for both bit states, we find

$$P_S = \tilde{p}_1 P_C + \tilde{p}_2 = e^{-r_0}/(p_1 e^{r_0} + p_2 e^{-r_0}). \tag{31}$$

The derivation for $r_0 < 0$ is similar and leads to the extra factor e^{2r_0}, so that the crossing probability in both cases can be written as $P_S = e^{-|r_0|}/(p_1 e^{r_0} + p_2 e^{-r_0})$.

Thus, using the trick of reducing the quantum dynamics to the classical problem, we have found the probability of successful uncollapsing for a DQD qubit with no Hamiltonian evolution:

$$P_S = \frac{e^{-|r_0|}}{e^{r_0}\rho_{in,11} + e^{-r_0}\rho_{in,22}}, \tag{32}$$

where ρ_{in} characterizes the qubit state before the first measurement. We will discuss this result in the next subsection. Before that let us rederive it in a different way, using the power of the standard methods of first passage theory.[28] This method has recently been used to investigate entanglement dynamics of jointly measured qubits.[29]

It is convenient to scale time in units of the measurement time, $\tau \equiv t/T_M$; then the probability distributions (25) take the simple form

$$P_{1,2}(r,\tau) = \sqrt{\frac{1}{2\pi\tau}}\exp\left(-\frac{(r\mp\tau)^2}{2\tau}\right). \tag{33}$$

These are the solutions of two different classical random walks with dimensionless drift velocity $\tilde{v}_{1,2} = \pm 1$ and dimensionless diffusion coefficient $\tilde{D} = 1/2$ described by the Fokker-Planck equations,

$$\partial_\tau P_i(r,\tau) = -\tilde{v}_i \partial_r P_i + \tilde{D}\partial_r^2 P_i. \tag{34}$$

In order to solve the first passage problem, we solve first for the Green functions $G(r,\tau)$ of the above equations starting from the initial condition $r = r_0$. The solutions from the different drift velocities will be weighted with probabilities $\tilde{p}_{1,2}$. These Green function equations are supplemented with an absorbing boundary condition at the origin ($r = 0$),

$$G_i(r = 0, \tau) = 0, \tag{35}$$

in order to account for the statistics of events that cross this point at least one time. Let as again start with assuming $r_0 > 0$ and consider the other case later. The solution of Eq. (34) subject to the condition (35) is most easily found by guessing:

$$\begin{aligned} G_i(r,\tau) \;&=\; \frac{1}{\sqrt{4\pi\tilde{D}\tau}}\left(\exp\left[-\frac{(r-r_0-\tilde{v}_i\tau)^2}{4\tilde{D}\tau}\right]\right. \\ &-\; \left.\exp\left[-\tilde{v}_i r_0/\tilde{D}\right]\exp\left[-\frac{(r+r_0-\tilde{v}_i\tau)^2}{4\tilde{D}\tau}\right]\right). \end{aligned} \tag{36}$$

In the form written above, it is obvious that the solution obeys the equation of motion (34) and has the correct initial condition r_0 at $\tau = 0$ (because the absorbing boundary condition only permits $r \geq 0$ solutions). Further inspection of the solution is facilitated by factoring out the free Green function, $G_{\rm free}(r,\tau) = \exp[-(r - r_0 - \tilde{v}_i\tau)^2/(4\tilde{D}\tau)]/\sqrt{4\pi\tilde{D}\tau}$ to write the solution as

$$G_i(r,\tau) = G_{\rm free}(r,\tau)\left[1 - \exp\left(-\frac{r\,r_0}{\tilde{D}\tau}\right)\right]. \tag{37}$$

One can now explicitly see that the absorbing boundary condition (35) is satisfied, and the solution is completely positive (as it must be to represent a probability density).

To calculate the first passage time distribution, we first note that the total survival probability that the random walker will be in the interval $r \in (0,\infty)$ at time τ is given by $P_{\rm sur}(\tau) = \int_0^\infty dr G(r,\tau)$. However, the only place for the particle to be lost from the system is at the origin. Therefore, the first passage time distribution $P^{(i)}_{\rm fpt}$ is given by

$$P^{(i)}_{\rm fpt} = -\partial_\tau P_{\rm sur} = -\int_0^\infty dr \partial_t G_i(r,\tau). \tag{38}$$

The next step is to note that the Fokker-Planck equation (34) may be rewritten as a continuity equation, $\partial_\tau G_i + \partial_r J_i = 0$. This simply means that locally, probability is conserved. The probability current in the continuity equation is $J_i = -\tilde{D}\partial_r G_i + \tilde{v}_i G_i$ from (34). Substituting this into (38) we find the general result

$$P^{(i)}_{\rm fpt}(\tau) = \int_0^\infty dr \partial_r J_i = J_i(\infty) - J_i(0) = -J_i(0), \tag{39}$$

because the probability current at infinity vanishes. Applied to our problem, we find

$$P^{(i)}_{\rm fpt}(\tau) = \frac{r_0}{\sqrt{4\pi\tilde{D}\tau^3}}\exp\left[-(r_0+\tilde{v}_i\tau)^2/(4\tilde{D}\tau)\right]. \tag{40}$$

The probability P_C that the point $r = 0$ is ever crossed is found by integrating (40) over all positive time to obtain

$$P_C = \begin{cases} \exp(-\tilde{v}_1 r_0/\tilde{D}) = \exp(-2r_0), & i = 1, \\ 1, & i = 2. \end{cases} \quad (41)$$

This result may be understood intuitively because if the state is in $i = 2$ then the drift $\tilde{v}_2 = -1$ causes $r(t)$ to evolve from r_0 to $-\infty$ and therefore must cross 0 at some time, while if the system is in state $i = 1$ then the drift $\tilde{v}_1 = 1$ causes $r(t)$ to evolve from r_0 to $+\infty$. Therefore, in order to cross $r = 0$, the noise term must fight against the drift, causing a successful crossing only occasionally.

In order to obtain the normalized first passage distribution (conditioned on crossing), we divide (40) by the probabilities (41) to obtain

$$P^{(i)}_{\text{fpt}}(\tau|C) = \frac{r_0}{\sqrt{4\pi\tilde{D}\tau^3}} \exp\left[-(r_0 - |\tilde{v}_i|\tau)^2/(4\tilde{D}\tau)\right]. \quad (42)$$

The mean first passage time may also be calculated from (42) to obtain $\tau_{c,i} = r_0/|\tilde{v}_i| = r_0$.

Obtaining analogous results for $r_0 < 0$ is straightforward because the Green function for the Fokker-Planck equation (34) is invariant under the transformation $\{r \to -r,\ r_0 \to -r_0,\ \tilde{v}_i \to -\tilde{v}_i$ (or $1 \leftrightarrow 2)\}$ which is also reflection symmetry about the origin. Therefore results (40,41,42) can be extended using this symmetry. Combining results, we can now calculate the total uncollapsing probability, $P_S = \tilde{p}_1 P_{C,1} + \tilde{p}_2 P_{C,2}$ to obtain the result (32) in this new, more powerful way.

C. Discussion

We now discuss the physical meaning of the result (32). When the first measurement result indicates a particular qubit state with good confidence ($|r_0| \gg 1$), the probability of success P_S given by Eq. (32) becomes very small, eventually becoming $P_S = 0$ for a projective measurement, realized for $r_0 = \pm\infty$. This recovers the traditional statement of irreversibility. In the other limit of $r_0 = 0$, the success probability is unity because no time needs to elapse - the state is already undisturbed. We stress that the possibility of uncollapsing as well as our formalism requires a quantum-limited detector, i.e. one that introduces no additional dephasing to the system. For such a detector measuring a pure state, the state remains pure throughout the partial collapse, and the uncollapse. We also note that if the qubit is entangled with other qubits, the uncollapsing restores the state of the whole system.

Let us compare the general upper bound (13) for the success probability P_S with the result (32). Substituting the probabilities in the bound (13) with probability densities, we find

$$P_S \leq \frac{\min\{P_1(\bar{I}), P_2(\bar{I})\}}{P_1(\bar{I})\rho_{in,11} + P_2(\bar{I})\rho_{in,22}}, \quad (43)$$

where $\bar{I}$ corresponds to the measurement result r_0. The bound coincides with Eq. (32) because $P_1(\bar{I})/P_2(\bar{I}) = e^{2r_0}$. This means that the "wait and stop" strategy analyzed above is optimal in the sense that it reaches the upper bound.

It is also instructive to not specify the result of the first measurement, but to find the total probability $\tilde{P}_S^{\text{total}}$ [see Eq. (20)] that the initial qubit state can be restored after a measurement for time t. This is given by averaging P_S in Eq. (32) over the results r_0 with the corresponding weights (28). This averaging is technically easier using the form of Eq. (43) and gives

$$\tilde{P}_S^{\text{total}} = \int d\bar{I} \min\{P_1(\bar{I}), P_2(\bar{I})\} = 1 - \text{erf}\left(\sqrt{\frac{t}{2T_M}}\right), \quad (44)$$

which depends only on the "strength" t/T_M of the first measurement, but not on the initial state, as expected from the discussion in Sec. II D. Notice that the result (44) reaches the upper bound (20) because (32) reaches the upper bound (13).

D. Statistical features of the time to uncollapse

In addition to the probability of success, the complete solution of the first-passage problem given above now allows us to specify further information about the uncollapsing process. In particular, an important question for an experimental implementation of this idea is how long it is necessary to wait.

Since the distribution of the first passage time (42) does not depend on the bit state, it directly gives the distribution of the waiting time to uncollapse any qubit state. Therefore, rescaling back the time axis in Eq. (42), we find the waiting time distribution is

$$P_{\text{wait}}(t) = \frac{|r_0|}{\sqrt{2\pi t^3/T_M}} \exp\left[\frac{-(|r_0| - t/T_M)^2}{2t/T_M}\right]. \quad (45)$$

This distribution is normalized, since we consider only successful attempts of uncollapsing. The fact that the distribution is independent of the initial qubit state is not surprising, since otherwise a successful uncollapsing instance would give us an information about the qubit state (see discussion in Sec. II D).

Using the distribution (45), we can find the mean waiting time to uncollapse

$$T_{\text{wait}} = T_M\,|r_0|, \quad (46)$$

the standard deviation

$$\Delta T_{\text{wait}} = T_M \sqrt{|r_0|}, \quad (47)$$

and the most likely waiting time (which maximizes P_{wait})

$$T_l = T_M \left(\sqrt{r_0^2 + 9/4} - 3/2\right). \quad (48)$$

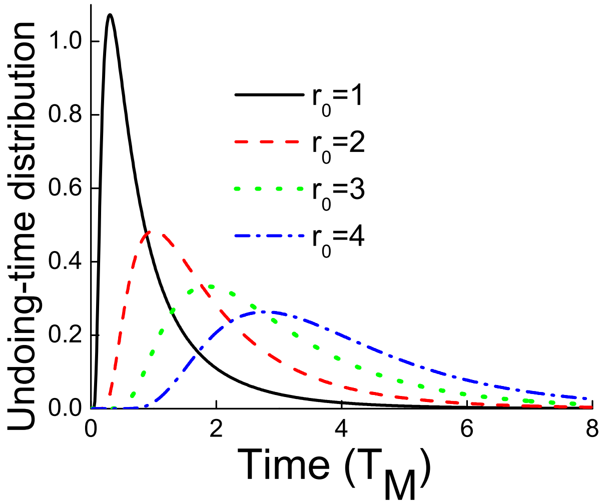

FIG. 2: (Color online). Probability distribution of the time required to undo the measurement. Different plots are for different values of r_0.

The distribution (45) of the waiting times is plotted in Fig. 2 for several values of r_0. Note that it has a long tail, which makes the average value $T_{\rm wait}$ to be longer than the most likely value T_l.

E. Evolving charge qubit

Let us now extend the first example of a single charge qubit measured by QPC, by including internal evolution of the qubit via a qubit Hamiltonian,

$$H_{QB} = -(\varepsilon/2)\sigma_z + H\sigma_x, \tag{49}$$

where ε is the energy asymmetry between the quantum dot levels, and H is the tunnel coupling between the dots. In this case Eqs. (22)–(24) are no longer valid and should be replaced by the Bayesian equations[24] (in Stratonovich form[25])

$$\dot\rho_{11} = -\dot\rho_{22} = -2\,H\,{\rm Im}\,\rho_{12} + \rho_{11}\rho_{22}\,\frac{2\Delta I}{S_I}\,[I(t)-I_0], \tag{50}$$

$$\dot\rho_{12} = i\varepsilon\rho_{12} + iH(\rho_{11}-\rho_{22}) - (\rho_{11}-\rho_{22})\frac{\Delta I}{S_I}\,[I(t)-I_0]\,\rho_{12}, \tag{51}$$

where $I(t)$ is the QPC current,

$$I(t) = \rho_{11}(t)I_1 + \rho_{22}(t)I_2 + \xi(t), \tag{52}$$

containing white noise $\xi(t)$ with spectral density S_I, and we use $\hbar=1$. These evolution equations are nonlinear and not very simple to deal with. To discuss the undoing of a continuous measurement, it is more convenient to use a non-normalized density matrix σ, which has an advantage of dealing with linear equations.

We rewrite Eqs. (50)–(51) in the form

$$\rho = \sigma/{\rm Tr}\sigma, \tag{53}$$

$$\dot\sigma_{11} = -2H\,{\rm Im}\sigma_{12} - \sigma_{11}\frac{1}{S_I}[I(t)-I_1]^2, \tag{54}$$

$$\dot\sigma_{22} = 2H\,{\rm Im}\sigma_{12} - \sigma_{22}\frac{1}{S_I}[I(t)-I_2]^2, \tag{55}$$

$$\dot\sigma_{12} = i\varepsilon\sigma_{12} + iH(\sigma_{11}-\sigma_{22}) - \sigma_{12}\left\{\frac{[I(t)-I_0]^2}{S_I} + \frac{(\Delta I)^2}{4S_I}\right\}, \tag{56}$$

so that $\sigma(0)=\rho(0)$, while the ratio $\sigma(t)/\rho(t)$ decreases with time and is equal to the normalized probability density of the corresponding realization of the detector output $I(t')$, $0\le t'\le t$.[30] In the language of general quantum measurement this formulation corresponds to omitting the denominator in Eq. (9). Notice that we still consider an ideal detector, so an initially pure state remains pure, $|\sigma_{12}|^2=\sigma_{11}\sigma_{22}$.[30]

A casual inspection of Eqs. (54)-(56) shows that they are seemingly not well-defined because the terms $[I(t)-I_{0,1,2}]^2$ contain the term $\xi(t)^2=\infty$ [from the relation $\langle\xi(t)\xi(0)\rangle=(S_I/2)\delta(t)$]. This divergence is artificial because there will always be a small correlation time T of the noise and/or a finite detector bandwidth B (corresponding to $T=1/4B$), so there will be a large but finite constant $\mathcal{C}=\langle\xi(t)^2\rangle=S_I/4T$ contained in terms of the form $[I(t)-I_{0,1,2}]^2$. It is easy to see that Eqs. (54)-(56) do not change if we subtract the same constant from these terms $[I(t)-I_{0,1,2}]^2\to[I(t)-I_{0,1,2}]^2-\mathcal{C}$. This can be shown by considering another unnormalized density matrix $\eta=\sigma\exp(t/T)$. Writing the linear Bayesian equations (54)-(56) in the form $\dot\sigma_{ij}=f_{ij}[\sigma]$, the equations transform to $\dot\eta_{ij}=f_{ij}[\eta\exp(-t/T)]\,\exp(t/T)+\eta_{ij}/T$ under the change of variables. The unnormalized Bayesian equations are linear in the density matrix elements σ_{ij}, so the exponential factors cancel out. The new equations are thus the same as the old ones with a constant $\mathcal{C}=S_I/4T$ subtracted from the $[I(t)-I_{0,1,2}]^2$ terms. The unspecified constant T in the density matrix transformation may be chosen to be the short correlation time T discussed above, thus canceling the large term and making Eqs. (54)-(56) well-defined. The only price to be paid for this transformation is an altered normalization, that will cancel in the normalized density matrix (53).

For a particular realization of the detector output $I(t')$, $0\le t'\le t$, Eqs. (54)-(56) define a linear map $\sigma(0)\to\sigma(t)$, corresponding to a particular Kraus operator M_m (which, therefore, can be denoted as $M_{\{I\}}$). For the uncollapsing we have to realize the map, corresponding to the inverse Kraus operator $CM_{\{I\}}^{-1}$ (see Sec. II). It is obvious that in contrast to the case of the non-evolving qubit, this cannot be done by simply continuing the measurement and waiting for a specific result. The reason is that now the map is characterized by 6 real parameters (8 parameters for a linear operator $CM_{\{I\}}^{-1}$ with neglected overall phase and normalization), instead

of 1 parameter for the non-evolving case [see Eq. (24) and (26)]. We will discuss a little later how the 6-parameter uncollapsing procedure can be realized explicitly. Before that we discuss how to find the operator $M_{\{I\}}$ in a more straightforward way, than from Eqs. (54)-(56).

Let us consider only the evolution of (unnormalized) pure states $|\psi(t)\rangle = \alpha(t)|1\rangle + \beta(t)|2\rangle$, so that $\sigma = |\psi\rangle\langle\psi|$. Then Eqs. (54)-(56) can be rewritten as

$$\dot{\alpha} = +i\frac{\varepsilon}{2}\alpha - iH\beta - \alpha\frac{1}{2S_I}[I(t) - I_1]^2, \tag{57}$$

$$\dot{\beta} = -i\frac{\varepsilon}{2}\beta - iH\alpha - \beta\frac{1}{2S_I}[I(t) - I_2]^2, \tag{58}$$

where the infinite part of $I^2(t)$ can be canceled in the same way as discussed above. The linearity of these equations guarantees that for any given realization of $I(t)$, it is sufficient to solve (57,58) for the initial states $|1\rangle$ and $|2\rangle$ in order to find the solution for an arbitrary initial state of the qubit. Defining $\vec{v}_1 = \alpha_1(t)|1\rangle + \beta_1(t)|2\rangle$ as the solution of (57,58) for initial state $|1\rangle$, and $\vec{v}_2 = \alpha_2(t)|1\rangle + \beta_2(t)|2\rangle$ as the solution of (57,58) for initial state $|2\rangle$, we can write the solution for an arbitrary initial state $|\psi_{in}\rangle = |\psi(0)\rangle = a|1\rangle + b|2\rangle$ as $|\psi(t)\rangle = a\vec{v}_1 + b\vec{v}_2$. Therefore, the Kraus operator $M_{\{I\}}$ for a given realization of $I(t)$, in the $|1\rangle, |2\rangle$ basis is

$$M_{\{I\}} = \begin{pmatrix} \alpha_1(t) & \alpha_2(t) \\ \beta_1(t) & \beta_2(t) \end{pmatrix}. \tag{59}$$

For the uncollapsing we need to apply the Kraus operator $CM^{-1}_{\{I\}}$, which maps the state $\vec{v}_1$ onto $C|1\rangle$ and the state $\vec{v}_2$ onto $C|2\rangle$. The reason why we need a non-unitary transformation is that the vectors $\vec{v}_1$ and $\vec{v}_2$ are in general non-orthogonal and have different norms. Geometrically, such a transformation can be done by using a relative shrinking or stretching of two orthogonal axes (found for a given $M_{\{I\}}$), which would make $\vec{v}_1$ and $\vec{v}_2$ orthogonal and equal in norm, followed by a unitary transformation (this would correspond to the decomposition of the form $U\sqrt{E}$ – see Sec. II B). However, for a practical realization of uncollapsing it is most natural to use the shrinking or stretching of the axes $|1\rangle$ and $|2\rangle$ via a continuous QND measurement with the QPC in the way considered above for a non-evolving qubit. In this case the uncollapsing procedure can be done in three steps (see Sec. II B): unitary evolution V, continuous QND measurement (where the qubit Hamiltonian is turned off, $\varepsilon = H = 0$) described by a diagonal matrix L, and a final unitary operation U. (In the notation of Sec. II B, V corresponds to $V_L^\dagger U_m^\dagger$, and U corresponds to $U_L^\dagger$.) These operators should satisfy

$$ULV = CM^{-1}_{\{I\}}, \tag{60}$$

and it is easy to find U, L, and V explicitly by recognizing Eq. (60) as a singular value decomposition of the operator $CM^{-1}_{\{I\}}$ (recall here that L is diagonal; also notice that the standard form for the singular value decomposition is slightly different, with V denoted as $V^\dagger$).

To find L explicitly, we notice that

$$C^{-2}M^\dagger_{\{I\}}M_{\{I\}} = UL^{-2}U^\dagger, \tag{61}$$

which is simply the diagonalization of $C^{-2}M^\dagger_{\{I\}}M_{\{I\}}$. Therefore,

$$L = C\begin{pmatrix} \lambda_-^{-1/2} & 0 \\ 0 & \lambda_+^{-1/2} \end{pmatrix} \quad \text{or} \quad L = C\begin{pmatrix} \lambda_+^{-1/2} & 0 \\ 0 & \lambda_-^{-1/2} \end{pmatrix}, \tag{62}$$

where

$$\lambda_\pm = \frac{||\vec{v}_1||^2 + ||\vec{v}_2||^2}{2} \pm \sqrt{\left(\frac{||\vec{v}_1||^2 - ||\vec{v}_2||^2}{2}\right)^2 + |\vec{v}_1\cdot\vec{v}_2^*|^2} \tag{63}$$

are the eigenvalues of the operator $M^\dagger_{\{I\}}M_{\{I\}}$ and the vectors $\vec{v}_i$ are defined above Eq. (59). The Cauchy-Schwartz inequality, $|\vec{v}_1\cdot\vec{v}_2^*|^2 \le ||\vec{v}_1||^2||\vec{v}_2||^2$, guarantees the non-negativity of λ_-. (The notation $\vec{v}_1\cdot\vec{v}_2^*$ is used for the inner product $\langle v_2|v_1\rangle$ of $\vec{v}_2$ and $\vec{v}_1$).

To find U, we use Eq. (61) again and see that the columns of U are composed of the eigenvectors of $M^\dagger_{\{I\}}M_{\{I\}}$ [the sequence of columns depends on the choice in Eq. (62)]. Finally, V is given by $V = U^\dagger CL^{-1}M^{-1}_{\{I\}}$. For brevity we will not show the matrices U and V explicitly.

In the physical realization of the uncollapsing procedure the measurement step L can be performed in exactly the same way as in Sec. III B. Comparing Eq. (62) with Eqs. (26) and (24), we see that the continuous measurement by the QPC should be stopped when the dimensionless measurement result $r(t)$ reaches the value

$$r_1 = \ln\sqrt{\lambda_+/\lambda_-} > 0 \quad \text{or} \quad r_2 = \ln\sqrt{\lambda_-/\lambda_+} < 0, \tag{64}$$

for the first and second choice in Eq. (62), respectively (the choice should be made beforehand, since it determines operation V). As previously mentioned, the constant C is not important here because the physical state is always normalized. The procedure fails if the desired result is not reached during the continuous measurement.

The unitary operations V and U can be practically realized in three substeps each: z-rotation on Bloch sphere by applying non-zero energy asymmetry ε for some time, y-rotation by applying non-zero tunneling H, and then one more z-rotation. However, the last z-rotation of V and the first z-rotation of U are simply added to each other (since L does not change the relative phase of the state components or, equivalently, the azimuth angle on the Bloch sphere). The corresponding trivial degree of freedom can be eliminated, for example, by realizing the operation V in only two substeps, without the second z-rotation.

Let us count the number of real parameters, characterizing the uncollapsing procedure. Since V and U together

provide $2 \times 3 - 1 = 5$ parameters, and the desired result r in the measurement step adds one more parameter, the overall number of parameters is 6. As expected, this is exactly the needed number of parameters characterizing an arbitrary Kraus operator for the qubit (neglecting normalization and overall phase). Let us also mention the fact from linear algebra that the singular value decomposition (60) is unique in the non-degenerate case ($\lambda_+ > \lambda_- > 0$), up to the permutation of singular values [corresponding to the choice in Eq. (62)] and arbitrary phase factors in columns of U, with compensating changes in V (this corresponds to the discussed above compensation of z-rotations).

Now let us discuss the probability P_S of the successful uncollapsing. From the general theory discussed in section II [Eq. (8)], it is bounded from above by a fraction, $P_S \leq \min P_{\{I\}}/P_{\{I\}}(|\psi_{in}\rangle)$, in which the denominator is the probability density of the given realization $I(t)$ for the initial state $|\psi_{in}\rangle = a|1\rangle + b|2\rangle$ (with $|a|^2 + |b|^2 = 1$), while the numerator is this probability minimized over all initial states. So, the denominator is given by the squared norm of the final state $|\psi(t)\rangle = a\vec{v}_1 + b\vec{v}_2$,

$$P_{\{I\}}(|\psi_{in}\rangle) = ||a\vec{v}_1 + b\vec{v}_2||^2, \tag{65}$$

while the numerator is given by minimizing (65) over all normalized initial states. It is easy to see that this minimum is equal to the minimum eigenvalue λ_- of the operator $M^\dagger_{\{I\}} M_{\{I\}}$, given by Eq. (63); therefore,

$$P_S \leq \frac{\lambda_-}{||a\vec{v}_1 + b\vec{v}_2||^2}. \tag{66}$$

Converting this result into the language of density matrices and simultaneously generalizing it to an arbitrary initial state ρ_{in}, we obtain the bound

$$P_S \leq \frac{\frac{||\vec{v}_1||^2+||\vec{v}_2||^2}{2} - \sqrt{\left(\frac{||\vec{v}_1||^2-||\vec{v}_2||^2}{2}\right)^2 + |\vec{v}_1 \cdot \vec{v}_2^*|^2}}{\rho_{in,11}||\vec{v}_1||^2 + \rho_{in,22}||\vec{v}_2||^2 + 2\mathrm{Re}[\rho_{in,12}\, \vec{v}_1 \cdot \vec{v}_2^*]}, \tag{67}$$

in which the numerator is the explicit expression (63) for λ_-. It is easy to check that this result reduces to the bound (43) in the non-evolving case, in which $\vec{v}_1 = \left(\sqrt{P_1(\bar{I})}, 0\right)^T$ and $\vec{v}_2 = \left(0, \sqrt{P_2(\bar{I})}\right)^T$.

The uncollapsing procedure discussed in this subsection is optimal in the sense that it corresponds to the upper bound of Eq. (67). To prove this statement, instead of calculating P_S explicitly, let us use the fact (see Sec. II D) that the product $P_S P_{\{I\}}$ cannot depend on the initial state. Therefore, it is sufficient to prove the optimality of P_S only for one initial state. Let us choose the state $|\psi_{in}\rangle$ that is the eigenvector of $M^\dagger_{\{I\}} M_{\{I\}}$, corresponding to the eigenvalue λ_-. Then after the first measurement (operator $M_{\{I\}}$) and the unitary operation V it is transformed into one of the basis states [$|1\rangle$ or $|2\rangle$ for the first or second choice in (62), respectively]. Recall that for the QND (non-evolving) measurement case, $r(t) \to \infty$ for the initial state $|1\rangle$, while $r(t) \to -\infty$ for the initial state $|2\rangle$. The crossing thresholds (64) indicate that the measurement L is always successful because $r(t)$ necessarily crosses the desired value (which is positive for $|1\rangle$ and negative for $|2\rangle$, as discussed above). Therefore, the uncollapsing success probability for this special state is 100%, that is equal to the upper bound (67). As mentioned above, the optimality of the procedure for this special state also proves the optimality for any initial state.

Obviously, an uncollapsing procedure can also be non-optimal. As an example, let us consider a procedure which realizes the desired mapping $\{\vec{v}_1, \vec{v}_2\} \to \{C|1\rangle, C|2\rangle\}$ using two measurements instead of one. The goal of the first measurement is to map $\{\vec{v}_1, \vec{v}_2\}$ into an orthogonal pair of vectors, while the goal of the second measurement is to equalize their norms, keeping them orthogonal. The first goal can be achieved by stretching/shrinking of the Hilbert space along any axis $\vec{u}$ of the form $\vec{v}_1 + c\vec{v}_2(\vec{v}_1 \cdot \vec{v}_2)/|\vec{v}_1 \cdot \vec{v}_2|$ with an arbitrary positive real number c (it is easy to visualize this procedure of making two vectors orthogonal by assuming the space of real vectors, for which the axis $\vec{u}$ is geometrically in between $\vec{v}_1$ and $\vec{v}_2$; the same geometrical idea works for complex vectors). We recall that measurement for a non-evolving qubit (Sec. III A) stretches (squeezes) the $|1\rangle$ axis, while squeezing (stretching) the $|2\rangle$ axis. Therefore, the first goal can be achieved by a unitary operation which rotates $\vec{u}$ into $|1\rangle$, followed by a continuous measurement (with a QPC) of a non-evolving qubit, to be stopped when the mapped vectors become orthogonal. After the vectors $\{\vec{v}_1, \vec{v}_2\}$ are transformed into an orthogonal pair by the (successful) first measurement, the second part of the procedure should stretch/shrink the 2D Hilbert space along the resulting vectors to make them equal in norm. This can be done similarly, by a unitary rotation and partial measurement of a non-evolving qubit. Finally, another unitary operation can be used to map the resulting pair of vectors into $\{C|1\rangle, C|2\rangle\}$, thus completing the uncollapsing procedure. Notice that both measurements are performed in the "wait and stop" manner, and both measurements should be successful to realize the uncollapsing. While the successfully uncollapsed state is still perfect in this procedure, the probability of success is lower than the bound (67). To prove this non-optimality, let us again use the initial eigenstate $|\psi_{in}\rangle$, which corresponds to eigenvalue λ_-, so that the bound (67) is 100%. Then the probability of success for the first measurement is in general less than 100% (it is 100% only for one specific axis discussed previously, while here we consider a range of possible axes by allowing c to vary). Thus, the success probability is less than 100% for this special state, and therefore P_S is below the bound (67) for any initial state.

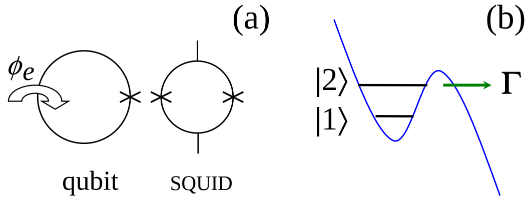

FIG. 3: (Color online). (a) Schematic of a phase qubit controlled by an external flux ϕ_e and inductively coupled to the detector SQUID. (b) Energy profile $V(\phi)$ with quantized levels representing qubit states. The tunneling event through the barrier is sensed by the SQUID.

IV. PHASE QUBIT

The second explicit example of erasing information and uncollapsing the wavefunction is for a superconducting phase qubit.[32] The system is comprised of a superconducting loop interrupted by one Josephson junction [Fig. 3(a)], which is controlled by an external flux ϕ_e in the loop. Two qubit states $|1\rangle$ and $|2\rangle$ [Fig. 3(b)] correspond to two lowest states in the quantum well for the potential energy $V(\phi)$ where ϕ is the superconducting phase difference across the junction. (Notice that the standard notation for the phase qubit states is $|0\rangle$ and $|1\rangle$;[32] however, we use $|1\rangle$ and $|2\rangle$ for consistency with the previous section.) The qubit is measured by lowering the barrier (which depends on ϕ_e), so that the upper state $|2\rangle$ tunnels into the continuum with the rate Γ, while the state $|1\rangle$ does not tunnel out. The tunneling event is sensed by a two-junction detector SQUID inductively coupled to the qubit [Fig. 3(a)]. Transitions between the levels $|1\rangle$ and $|2\rangle$ can be induced by applying microwave pulses that are resonant with the energy level difference.

A. Partial collapse

For sufficiently long tunneling time t, $\Gamma t \gg 1$, the measurement is a (partially destructive) projective measurement:[32] the system is destroyed if the tunneling occurs, while if there is no record of tunneling, then the state is projected onto the lower state $|1\rangle$. This measurement technique is remarkable in the fact that the wavefunction is collapsed *if nothing happens.* A more subtle situation arises if the barrier is raised after a finite time $t \sim \Gamma^{-1}$: then the measurement is only partial and therefore is of a POVM-type.[33] The system is still destroyed if tunneling happens, while in the case of no tunneling (which we refer to as a null-result measurement) the state is partially collapsed. This situation may be described with a two-outcome POVM, with elements E_n and E_y, where n denotes the null result, and y denotes the affirmative (tunneling) result. The POVM elements, given in the $|1\rangle$, $|2\rangle$ basis are

$$E_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\Gamma t} \end{pmatrix}, \qquad E_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 - e^{-\Gamma t} \end{pmatrix}, \tag{68}$$

with the obvious completeness relation $E_n + E_y = \mathbf{1}$.

It is interesting to notice that while $E_n = M_n^\dagger M_n$ corresponds to a Kraus operator M_n (see discussion below), no meaningful Kraus operator M_y can be introduced for the POVM element E_y, because in the case of a tunneling event the system leaves its two-dimensional Hilbert space and becomes incoherent (so that a single Kraus operator cannot be introduced even in the extended Hilbert space). However, this is not important for us because we are interested in the null-result case only.

Limiting the unitary operation in the decomposition (3) to be the phase factor only, we may expect the null-result Kraus operator M_n to be of the form $M_n = \mathrm{diag}\{1, e^{-\Gamma t/2} e^{-i\varphi}\}$. In a simple model[34,35] the phase φ is zero in the rotating frame, which compensates for the energy difference of the states $|1\rangle$ and $|2\rangle$. In a real experiment,[33] however, this energy difference changes in the process of measurement because it is affected by ϕ_e, and therefore even in the rotating frame the phase φ is non-zero. Correspondingly, the qubit density matrix changes after the null-result measurement as

$$\frac{\rho_{11}(t)}{\rho_{22}(t)} = \frac{\rho_{in,11}}{e^{-\Gamma t}\rho_{in,22}}, \quad \frac{\rho_{12}(t)}{\sqrt{\rho_{11}(t)\rho_{22}(t)}} = \frac{e^{i\varphi(t)}\rho_{in,12}}{\sqrt{\rho_{in,11}\rho_{in,22}}}. \tag{69}$$

In the real experiment[33] the situation is even more complex because the tunneling rate gradually changes in time; also, instead of controlling the measurement time t, it is much easier to control the tunneling rate. As a result, the measurement should be characterized by the overall strength $p_t = 1 - \exp[-\int_0^t \Gamma(t')dt']$. Nevertheless, for simplicity, we use here the physically transparent language of Eq. (69) with $e^{-\Gamma t}$ understood as $1 - p_t$.

Up to such changes of notation, the coherent non-unitary evolution (69) has been experimentally verified in Ref. 33 using tomography of the post-measurement state. The state tomography consisted of 3 types of rotations of the qubit Bloch sphere, followed by complete (projective) measurement. Actually, in the experiment it was not possible to select only the null-result cases, because it was not possible to distinguish if a tunneling event happened during measurement or during tomography. However, a simple trick of comparing the protocols with and without tomography made it possible to separate the null-result cases.

Notice that except for the effect of extra phase $\varphi(t)$, the qubit evolution (69) is similar to the qubit evolution in the example of Sec. III A; in particular, it also represents an ideal measurement which does not decohere the qubit. Formally, the evolution (69) corresponds to the measurement result $r = \Gamma t/2$ in Eq. (26). As will be shown later, the probability to undo the measurement is still given by Eq. (32) using this value of r.

B. Uncollapsing

We will now describe how to undo the state disturbance (69) caused by the partial collapse resulting from the null-result measurement. The undoing of this measurement consists of three steps: (i) Exchange the amplitudes for the states $|1\rangle$ and $|2\rangle$ by application of a microwave π-pulse, (ii) perform another measurement by lowering the barrier, identical to the first measurement, (iii) apply a second π-pulse. If the tunneling did not happen during the second measurement, then the information about the initial qubit state is canceled (both basis states have equal likelihood for two null-result measurements). Correspondingly, according to Eq. (69) (which is applied for the second time with exchanged indices $1 \leftrightarrow 2$), any initial qubit state is fully restored. An added benefit to this strategy is that the phase φ is also canceled automatically; the physics of this *phase cancellation* is the same as in the spin-echo technique for qubits.

It is easy to mistake the above pulse-sequence as simply the well known spin-echo technique alone. We stress that this is not the case: Spin-echo deterministically reverses an unknown unitary transformation (arising usually from a slowly varying magnetic field) without gaining or losing any information about what that state is. Our strategy is probabilistic and requires erasing the classical information that one extracts from the system to begin with. It is a (probabilistic) reversal of a known non-unitary transformation - and therefore quite different from spin echo.

The success probability P_S for the uncollapsing strategy described above may be calculated by noting that it is just the probability that the second measurement gives a null result. If we start with the qubit state ρ_{in}, the state after the first measurement is given by Eq. (69), and after the π-pulse the occupation of the upper state is $\tilde{\rho}_{22} = \rho_{in,11}/[\rho_{in,11} + \rho_{in,22}\, e^{-\Gamma t}]$. The success probability is simply the probability that the second tunneling will not occur, $P_S = \tilde{\rho}_{11} + \tilde{\rho}_{22} e^{-\Gamma t} = 1 - \tilde{\rho}_{22}(1 - e^{-\Gamma t})$, which can be expressed as

$$P_S = \frac{e^{-\Gamma t}}{\rho_{in,11} + e^{-\Gamma t}\rho_{in,22}}, \tag{70}$$

and formally coincides with Eq. (32) for $r = \Gamma t/2$. We can verify that this strategy is optimal by using the E_n POVM element (68) together with the general result (13) to find the upper bound for the success probability P_S. The numerator of (13) is the smallest eigenvalue of E_n, which is $\exp(-\Gamma t)$, while the denominator is $\mathrm{Tr} E_n \rho_{in} = \rho_{in,11} + e^{-\Gamma t}\rho_{in,22}$, giving a P_S that coincides with Eq. (70), and thus confirming the optimality of the analyzed uncollapsing procedure.

The total uncollapsing probability $\tilde{P}_S$ of two null results (see Sec. II D) is

$$\tilde{P}_S = (\rho_{in,11} + e^{-\Gamma t}\rho_{in,22})P_S = e^{-\Gamma t}. \tag{71}$$

As expected (see Sec. II D) this probability does not depend on the initial state.

Uncollapsing of the phase qubit state has recently been experimentally realized by Nadav Katz and colleagues in the lab of John Martinis, at UC Santa Barbara.[11] The experimental protocol was slightly shorter than that described above: it was missing the second π-pulse, so the uncollapsed state was actually the π-rotation of the initial state. Shortening of the protocol helped in decreasing the duration of the pulse sequence, which was about 45 ns, including the state tomography. Since the qubit energy relaxation and dephasing times were significantly longer, $T_1 = 450$ ns and $T_2^* = 350$ ns, the simple theory described above was sufficiently accurate. The same trick as for the partial-collapse experiment[33] was used to separate tunneling events during the first, second, and tomography measurements, because the detector SQUID was too slow to distinguish them directly.

The uncollapse procedure should restore any initial state. However, instead of examining all initial states to check this fact, it is sufficient to choose 4 initial states with linearly independent density matrices and use the linearity of quantum operations.[14] In the experiment[11] the uncollapse procedure was applied to the initial states $(|1\rangle + |2\rangle)/\sqrt{2}$, $(|1\rangle - i|2\rangle)/\sqrt{2}$, $|1\rangle$, and $|2\rangle$, and then the results were expressed via the language of the quantum process tomography[14,36] (QPT). The experimental[11] QPT fidelity of the uncollapsing procedure was above 70% for $p_t < 0.6$. A significant decrease of the uncollapsing fidelity for larger measurement strength p_t, especially for $p_t > 0.8$, was due to finite T_1 time and the fact that the null-result selection preferentially selects the cases with energy relaxation events, so that the procedure should no longer work well when $1 - p_t$ becomes comparable to the probability of energy relaxation.

As mentioned above, the uncollapsing procedure described in this subsection is theoretically optimal in the sense that it maximizes the bound (13) for the uncollapsing probability. An example of a non-optimal uncollapsing for a phase qubit was considered in Ref. 16. It was shown that if the measurement process is performed simultaneously with Rabi oscillations, then in the null-result case the initial state is periodically restored. The non-optimality of uncollapsing for such a procedure is due to measurement of an evolving qubit, which corresponds to a sequence of many measurements; a similar reason for the non-optimality of the two-step uncollapsing was discussed at the end of Sec. III E.

Exact uncollapsing requires an ideal detection, which does not decohere a quantum state; Eq. (69) corresponds to such an ideal detection. However, if various decoherence mechanisms are taken into account,[35] then only imperfect uncollapsing is possible. The theory of imperfect uncollapsing is a subject of further research.

V. GENERAL PROCEDURE FOR ENTANGLED CHARGE QUBITS

Let us present an explicit procedure which can be used in principle to undo an arbitrary measurement M_m of any number N of entangled qubits with maximum probability. For simplicity we consider double-quantum-dot charge qubits and assume that any unitary transformation can be used in the procedure. If the operator M_m was produced by a one-qubit measurement, and other entangled qubits were not experiencing a Hamiltonian evolution, then the formalism of Sec. III A is essentially unchanged,[38] and uncollapsing of the measured qubit leads to the restoration of the whole entangled state. If the operator M_m was produced by a one-qubit measurement, while other qubits were evolving in a unitary way but not interacting with the measured qubit, then uncollapsing is also easy: we should uncollapse the measured qubit in the usual way (Sec. III) and should apply inverse unitary transformation for other qubits. In this section, however, we do not consider these simple special cases; the goal is to undo an arbitrary Kraus operator M_m.

Let us decompose M_m as $M_m = U_m\sqrt{E_m}$ [see Eq. (3)]. Reversing the unitary operation U_m can be done in the regular Hamiltonian way, so the nontrivial part is undoing the $\sqrt{E_m}$ operator. We recall the diagonalization of E_m is given by $E_m = \sum_i p_i^{(m)}|i\rangle\langle i|$ with vectors $|i\rangle$ forming an orthonormal basis. As discussed in Sec. II B, for the optimal uncollapsing which maximizes the success probability, we have to perform a procedure corresponding to another measurement operator $\tilde{L} = \sqrt{\min_j p_j^{(m)}}\, E_m^{-1/2}$ which is also diagonal in the basis $|i\rangle$ with corresponding matrix elements $\tilde{L}_{ii} = \langle i|\tilde{L}|i\rangle = \sqrt{(\min_j p_j^{(m)})/p_i^{(m)}}$, all of which are between 0 and 1. Given N qubits, i ranges from 1 to 2^N. Notice that $\tilde{L}$ is obviously Hermitian.

Our procedure is to realize $\tilde{L}$ with a sequence of null-result measurements and unitary operations. Shown in Fig. 4 is an illustration of the physical set-up that is used for the measurements: a QPC (tunnel junction) capacitively coupled to N non-evolving DQD charge qubits. We assume that the QPC is tuned to a highly nonlinear regime, for which no electron can tunnel across the QPC barrier on experimentally relevant time-scales unless all qubits are in the state $|1\rangle$. We name this multi-qubit state $|\mathbb{1}\rangle \equiv |1, 1, \ldots, 1\rangle$. Such a regime is possible because of the exponential dependence of the tunneling rate on QPC barrier height, while the barrier height depends linearly on the states of the coupled qubits. Of course, this regime is not quite realistic; however, we discuss the procedure in principle. We also assume that even for the N-qubit state $|\mathbb{1}\rangle$, the rate γ of electron tunneling through the QPC is rather low, so that we can distinguish single tunneling events (technically, this would require an additional single-electron transistor). If we perform the measurement during time t and see no tunneling through the QPC, then similarly to the case of Sec. IV A, the corresponding null-result Kraus operator M_n shrinks the $|\mathbb{1}\rangle$ axis of the Hilbert space by the factor $e^{-\gamma t/2}$, while leaving all perpendicular axes unchanged. For the matrix elements this means $\langle\mathbb{1}|M_n|\mathbb{1}\rangle = e^{-\gamma t/2}$, $\langle\psi_j^{\perp}|M_n|\psi_{j'}^{\perp}\rangle = \delta_{jj'}$, $\langle\psi_j^{\perp}|M_n|\mathbb{1}\rangle = \langle\mathbb{1}|M_n|\psi_j^{\perp}\rangle = 0$, where we introduced a set of $2^N - 1$ states $|\psi_j^{\perp}\rangle$ spanning the subspace orthogonal to $|\mathbb{1}\rangle$.

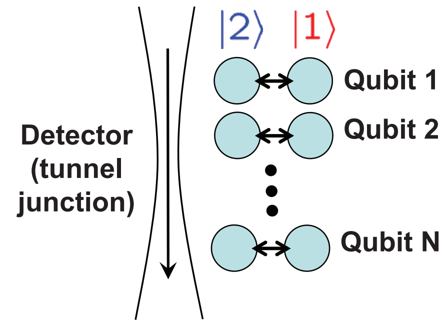

FIG. 4: (Color online.) Schematic set-up for uncollapsing of N entangled qubits. The tunnel junction detector (QPC) is in a strongly nonlinear regime, so that an electron can tunnel through it with rate γ only when all qubits are in state $|1\rangle$.

The general strategy to implement the operator $\tilde{L}$ is the following. We first note that in the basis $|i\rangle$ that diagonalizes $\tilde{L}$, this diagonal matrix can be represented as a product of 2^N diagonal matrices, where each term in the product has all diagonal entries as 1, except the ith entry: $\text{diag}\{1, 1, \ldots, \tilde{L}_{ii}, \ldots, 1\}$. Each of these matrices may be interpreted as a separate Kraus operator that can be sequentially implemented. Thus, the explicit physical procedure consists of 2^N steps, each of which has 3 substeps. First, we apply a unitary transformation U_1 which transforms the first basis vector $|i = 1\rangle$ into the state $|\mathbb{1}\rangle$. Then the evolution of all qubits is stopped, and the detector is turned on for a time t_1. This time is chosen so that the null-result Kraus operator $L^{(1)}$ has the desired matrix element $\langle\mathbb{1}|L^{(1)}|\mathbb{1}\rangle = \tilde{L}_{11}$; this condition yields $t_1 = -2\gamma^{-1}\ln\tilde{L}_{11}$. The measurement is then followed by the reverse unitary, $U_1^\dagger$, to take the state $|\mathbb{1}\rangle$ back to state $|i = 1\rangle$. This 3-substep procedure is then repeated for $i = 2, 3, \ldots, 2^N$, sequentially transforming the state $|i\rangle$ to $|\mathbb{1}\rangle$ with unitary U_i, and performing measurement with the detector for a time $t_i = -2\gamma^{-1}\ln\tilde{L}_{ii}$, followed by the reverse unitary, $U_i^\dagger$. This sequence of steps decomposes the uncollapsing operator $\tilde{L}$ as

$$\tilde{L} = U_{2^N}^\dagger L^{(2^N)} U_{2^N} \cdots U_2^\dagger L^{(2)} U_2 U_1^\dagger L^{(1)} U_1. \tag{72}$$

The uncollapsing procedure is successful only if there were no tunneling events in the QPC. By construction, the success probability P_S for this procedure maximizes the general bound (13).

The success probability P_S for the uncollapsing process $\rho_m \to \rho_{in}$ with $\rho_{in} = \tilde{L}\tilde{\rho}\tilde{L}^\dagger/\text{Tr}(\tilde{L}^\dagger\tilde{L}\tilde{\rho})$ and $\tilde{\rho} = U_m^\dagger\rho_m U_m$,

can be calculated as

$$P_S = \mathrm{Tr}(\tilde{L}^\dagger \tilde{L} \tilde{\rho}) = \sum_i \tilde{L}_{ii}^2 \tilde{\rho}_{ii} = \sum_i \tilde{\rho}_{ii} \exp(-\gamma t_i), \quad (73)$$

where $\tilde{\rho}_{ij}$ are the matrix elements of $\tilde{\rho}$ in the basis $|i\rangle$, which diagonalizes $\tilde{L}$. We may also find this result from another perspective by realizing that the success probability is simply the product of the null-result probabilities $p_S^{(i)}$ of all 2^N measurements,

$$P_S = \prod_i^{2^N} p_S^{(i)}, \;\; p_S^{(i)} = \frac{\sum_{j=1}^{i} \tilde{\rho}_{jj} e^{-\gamma t_j} + \sum_{j=i+1}^{2^N} \tilde{\rho}_{jj}}{\sum_{j=1}^{i-1} \tilde{\rho}_{jj} e^{-\gamma t_j} + \sum_{j=i}^{2^N} \tilde{\rho}_{jj}}, \quad (74)$$

where the expression for $p_S^{(i)}$ comes from comparing the traces of unnormalized density matrices after each of 2^N steps of the procedure. It is instructive to show explicitly that this expression for $p_S^{(i)}$ is equal to the expected expression

$$p_S^{(i)} = 1 - (1 - e^{-\gamma t_i}) \tilde{\rho}_{ii}^{(i)}, \quad (75)$$

in which $\tilde{\rho}^{(i)}$ is the normalized density matrix before the ith step of the procedure (after $i - 1$ null-result steps). This can be done if we prove the relation

$$\prod_{i=1}^{k} [1 - \tilde{\rho}_{ii}^{(i)} (1 - e^{-\gamma t_i})] = 1 - \sum_{i=1}^{k} (1 - e^{-\gamma t_i}) \tilde{\rho}_{ii}, \quad (76)$$

[notice that the right-hand-side of this equation is equal to the numerator in Eq. (74) with substitution $k \to i$]. Equation (76) can be proven by induction using the relation $\tilde{\rho}_{ii}^{(i)} = \tilde{\rho}_{ii} / \prod_{j=1}^{i-1} [1 - (1 - e^{-\gamma t_j}) \tilde{\rho}_{jj}^{(j)}]$, which can be easily derived recursively, $\tilde{\rho}_{ii}^{(j)} \to \tilde{\rho}_{ii}^{(j+1)}$, starting from $\tilde{\rho}_{ii}^{(1)} = \tilde{\rho}_{ii}$. In this way we show consistency between the null-result probabilities given by Eqs. (74) and (75), permitting the calculation of P_S in two independent ways.

Let us mention again that the uncollapsing procedure considered in this section reaches the upper bound (13) for the success probability P_S, that can be seen both by construction and explicitly.

VI. RECENT DEVELOPMENTS IN WAVEFUNCTION UNCOLLAPSE

Before concluding, we wish to give a summary of some interesting recent developments in this area of research. We will briefly discuss one theory proposal and one experiment.

A. Spin qubit

The examples given above mainly concern quantum dot charge qubits. It is a natural question if a similar kind of partial collapse/uncollapse can be carried over to spin qubits. An analysis of this situation was carried out by Trauzettel, Burkard, and one of the authors.[39] There it was shown how an uncollapse measurement can be realized using a scheme similar to the recent experiments by Koppens *et al.*[40] The essential idea of the spin-qubit experiments[40,41,42,43] is to manipulate and measure the spin of a single electron through the charge degree of freedom. This technique circumvents the otherwise difficult problem of controlling the weakly interacting spin. While we refer the reader to Refs. 39,40,41,42,43 for the details, we will give a simplified thumb-nail sketch of the physics here.

The qubit is encoded with two electron spins, where each electron is confined in a separate quantum dot. In contrast to our charge qubit discussion, these dots are open, with electrons able to enter and leave. Electrical bias is applied across this double quantum dot leading to charge transport. Electrons can tunnel sequentially, but *spin blockade*[44] restricts transport to situations where the two electrons form a spin singlet $(0, 2)S$ on the right dot while the spin triplet $(0, 2)T$ is outside the transport energy window due to the large single quantum dot exchange energy [here (n, m) refers to n electrons on the left dot and m electrons on the right dot]. This blockade physics provides an interesting initialization procedure of the quantum register - when the single-electron current stops flowing, we are confident that the two-electron state is in a $(1, 1)T$ state, because in the absence of spin flip processes, the tunneling transition to the $(0, 2)$ state is forbidden. From this configuration, it is possible to manipulate the system by applying electron spin resonance pulses,[40] transitioning the state to have overlap with the singlet state. Thus, the electron on the left dot may tunnel (with rate Γ) to the right dot and exit the system, giving rise to a small electrical current at the drain when this process is repeated many times. Of course, this will happen with some probability controlled by the overlap of the state with the singlet.

Drawing on our experience with the phase qubit, it is clear how to devise a weak measurement experiment and an uncollapsing experiment: the allowed transition can be permitted for a time of one's choosing and then forbidden by detuning the energy levels with a voltage pulse to one of the quantum dot's gates. In this way one can weakly probe the two-electron state, and in the null-result case (no single electron tunneling) partially collapse it to the triplet subspace. In order to propose the uncollapse part of the experiment, it is easiest to consider the case when the nuclear spins quickly admixed the singlet state with a triplet state, permitting the two-qubit state to encode one effective qubit: parallel or anti-parallel spins. The weak measurement technique described above will then partially collapse the state toward the parallel state under a null-measurement (no single electron tunneling). If now a π-pulse is applied to one of the spins with electron spin resonance, followed by a second null-measurement, this was shown to uncollapse

the state of the effective qubit.[39]

B. Optical polarization qubit

Another interesting development is the experimental implementation of wavefunction uncollapse for optical qubits using the polarization degree of freedom of single photons by Kim *et al.*[45] The weak measurement was implemented by passing the photon through a glass plate oriented at the Brewster angle. Only the vertical polarization is reflected off of the glass plate (with some probability). By placing a single-photon detector where the photon would have gone had it reflected, a null-click measurement partially collapses the polarization state to the horizontal polarization. The strength of the measurement can be increased by placing a series of plates in a row, effectively increasing the net probability of a vertically-polarized photon reflecting at some point.

The wavefunction uncollapse is done by inserting a half-wave plate (exchanging the amplitude of horizontal with vertical polarization), and having the same number of plates traversed by the photon again. If none of the single-photon detectors click, the polarization state is uncollapsed. This has been verified[45] with quantum state tomography (with polarizer and single-photon counter placed after all of the reflecting plates) on the photon, conditioned on none of the other photon detectors firing. The experiment showed an uncollapsing fidelity of above 94% for measurement strengths up to 0.9. It was also pointed out that the information from the first weak measurement can be used for developing guessing strategies about the unknown initial state. Two such strategies were presented, and one was shown to be optimal. Of course, in the case where the measurement was subsequently undone, these strategies did no better than random guessing.

VII. CONCLUSION

We have reviewed and extended recent developments in the theory (and experiment) of wavefunction uncollapse by undoing quantum measurements. We have formulated the problem of wavefunction uncollapse in terms of a contest between the uncollapse proponent, Plato, and the uncollapse skeptic, Socrates, monitored by the arbiter, Aristotle. Plato claims to have the ability to uncollapse wavefunctions, and this ability can be tested under the rules of the contest set forth.

We have discussed several general features of the uncollapse process in the abstract case, such as the upper bound on the success probability and quantum information aspects of the problem. In order to probabilistically undo the measurement, it is necessary to erase the information extracted about the state in the first measurement. This is a necessary condition to uncollapse the wavefunction, because otherwise various paradoxes arise. However, the information erasure is surely not a sufficient condition: the unitary evolution should also be properly reversed and, as the most experimentally challenging condition, the process should not bring decoherence, which requires a very good (ideal) detector.

In addition to discussing the theory of wavefunction uncollapse in the abstract case, we have also considered a variety of solid-state implementations and specific practical strategies for wavefunction uncollapse. Not only the success probability, these specific systems also allow the calculation of other characteristics of the process, such as the waiting time distribution for wavefunction uncollapse in the charge qubit case. The cases of the charge qubit (with and without Hamiltonian dynamics), the phase qubit, and many entangled charge qubits have been examined in detail. Additionally, we have also discussed two experimental realizations of this physics, based on the phase qubit and the polarization qubit, both of which have clearly demonstrated wavefunction uncollapse with high fidelity.

The ideality of the detector is necessary for perfect uncollapsing, and we have only dealt with these kinds of detectors in the theory section of this paper (by this we mean the detector adds no extra decoherence to the system). If a detector is slightly non-ideal, then even a perfectly executed uncollapse strategy will result in a slight infidelity in the final state. This is indeed the case in the experiments mentioned above although the fidelity was quite high. In such a situation there are two characteristics to contend with: the fidelity of uncollapsing as well as the probability of claimed success. It is an open topic for future research how these characteristics are related.

Acknowledgments

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[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, 1955).
[2] J. A. Wheeler, *Law Without Law*, in *Quantum Theory and Measurement*, Edited by J. A. Wheeler and W. H. Zurek, (Princeton Series in Physics, Princeton University Press, 1983).
[3] E. B. Davies, *Quantum Theory of Open Systems* (Academic, London, 1976); K. Kraus, *States, Effects, and*

Operations: Fundamental Notions of Quantum Theory (Springer, Berlin, 1983); C. M. Caves, Phys. Rev. D **33**, 1643 (1986); H. J. Carmichael, *An Open System Approach to Quantum Optics*, Lecture notes in physics (Springer, Berlin, 1993);M. B. Mensky, Phys. Usp. **41**, 923 (1998); A. S. Holevo, *Statistical Structure of Quantum theory* (Springer, 2001); D. F. Walls and G. J. Milburn, *Quantum Optics*, (Springer, 2006).

[4] A. N. Korotkov and A. N. Jordan, Phys. Rev. Lett. **97**, 166805 (2006).

[5] A. N. Jordan and A. N. Korotkov, in *Coherence and Quantum Optics IX*, edited by N. P. Bigelow et al. (Optical Soc. of America, 2008), p. 191.

[6] M. Koashi and M. Ueda, Phys. Rev. Lett. **82**, 2598 (1999).

[7] A. Ban, J. Phys. A **34**, 9669 (2001); G.M. D'Ariano, Fortschr. Phys. **51**, 318 (2003); M. A. Nielsen and C. M. Caves, Phys. Rev. A **55**, 2547 (1997); A. Royer, Phys. Rev. Lett. **73**, 913 (1994); **74**, 1040(E) (1995); M. Ueda, N. Imoto, and H. Nagaoka, Phys. Rev. A **53**, 3808 (1996); H. Mabuchi and P. Zoller, Phys. Rev. Lett. **76**, 3108 (1996); F. Buscemi, M. Hayashi, M. Horodecki, Phys. Rev. Lett. **100**, 210504 (2008).

[8] For various perspectives on these developments, see A. Gefter, New Scientist **194**(2603), 32 (2007); Z. Merali, Nature (London) **454**, 8 (2008); C. Bruder and D. Loss, Physics **1**, 34 (2008).

[9] M. O. Scully and K. Drühl, Phys. Rev. A **25**, 2208 (1982).

[10] A. Peres, *Quantum Theory: Concepts and Methods*, (Kluwer Academic Publishers, Dordrecht, 1995), p. 373.

[11] N. Katz, M. Neeley, M. Ansmann, R. C. Bialczak, M. Hofheinz, E. Lucero, A. O'Connell, H. Wang, A. N. Cleland, J. M. Martinis and A. N. Korotkov, Phys. Rev. Lett. **101**, 200401 (2008).

[12] One of the possible procedures is the following. If the post-measured state is not pure, we can apply more measurements to make it pure, and then probabilistically apply a unitary operation from an easily calculable set, which creates a mixture of pure states identical to the initial state.

[13] An exception is when the initial state belongs to a very limited set, in which the measurement result corresponds to only one state.

[14] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000).

[15] C. W. Gardiner and P. Zoller, *Quantum noise* (Springer, Berlin, 2004), Ch. 2.2.3.

[16] R. Ruskov, A. Mizel, and A. N. Korotkov, Phys. Rev. B **75**, 220501(R) (2007).

[17] In this example, Plato could change his undoing strategy to simply apply a tailored unitary to shift the disturbed state $|\psi_{1,m}\rangle$ back to its original state $|\psi_1\rangle$. Under this modified strategy, the success probability will be $\mathcal{P}$, which may or may not exceed P_S, depending on the strength of the measurement. However, in this case, Plato himself will not know whether the strategy succeeded or not, and therefore cannot claim to have an uncollapse procedure.

[18] E. T. Jaynes, *Probability Theory: The Logic of Science*, (Cambridge University Press, 2003).

[19] W. Feller, *An Introduction to Probability Theory and Its Applications*, v. 1 (Wiley, NY, 1968).

[20] A. Royer, Phys. Rev. Lett. **73**, 913 (1994); **74**, 1040(E) (1995).

[21] S. A. Gurvitz, Phys. Rev. B **56**, 15215 (1997); A. N. Korotkov and D. V. Averin, Phys. Rev. B **64**, 165310 (2001); H. S. Goan and G. J. Milburn, Phys. Rev. B **64**, 235307 (2001); S. Pilgram and M. Büttiker, Phys. Rev. Lett. **89**, 200401 (2002); A. A. Clerk, S. M. Girvin, and A. D. Stone, Phys. Rev. B **67**, 165324 (2003); D. V. Averin, in "Exploring the Quantum-Classical Frontier: Recent Advances in Macroscopic and Mesoscopic Quantum Phenomena", Eds. J. R. Friedman and S. Han (Nova Science, Huntington, NY, 2003); cond-mat/0004364; A. Shnirman, D. Mozyrsky, and I. Martin, Europhys. Lett. **67**, 840 (2004); A. A. Clerk and A. D. Stone, Phys. Rev. B **69**, 245303 (2004); A. N. Jordan and M. Büttiker, Phys. Rev. Lett. **95**, 220401 (2005); N. P. Oxtoby, P. Warszawski, H. M. Wiseman, H. B. Sun, R.E.S. Polkinghorne, Phys Rev B **71**, 165317 (2005); A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, arXiv:0810.4729v1.

[22] T. Hayashi, T. Fujisawa, H.-Du Cheong, Y.-Ha Jeong, and Y. Hirayama, Phys. Rev. Lett. **91**, 226804 (2003); J.R. Petta, A.C. Johnson, C.M. Marcus, M.P. Hanson, and A.C. Gossard, Phys. Rev. Lett. **93**, 186802 (2004); J.R. Petta, A.C. Johnson, J.M. Taylor, E.A. Laird, A. Yacoby, M.D. Lukin, C.M. Marcus, M. P. Hanson, and A.C. Gossard, Science **309**, 2180 (2005); J. M. Elzerman, R. Hanson, J. S. Greidanus, L. H. Willems van Beveren, S. De Franceschi, L. M. K. Vandersypen, S. Tarucha, and L. P. Kouwenhoven Phys. Rev. B **67**, 161308(R) (2003); A. K. Hüttel, S. Ludwig, K. Eberl, J. P. Kotthaus, Phys. Rev. B **72**, R081310 (2005); J. Gorman, E. G. Emiroglu, D.G. Hasko, and D.A. Williams, Phys. Rev. Lett. **95**, 090502 (2005).

[23] We use normalization of the shot noise, in which $S_I = 2eI(1-\mathcal{T})$, where $\mathcal{T}$ is the QPC transparency.

[24] A. N. Korotkov, Phys. Rev. B **60**, 5737 (1999); **63**, 115403 (2001); **67**, 235408 (2003).

[25] B. Øksendal, *Stochastic differential equations* (Springer, Berlin, 1998).

[26] A. N. Jordan and M. Büttiker, Phys. Rev. B **71**, 125333 (2005); A. N. Jordan, A. N. Korotkov, and M. Büttiker, Phys. Rev. Lett. **97**, 026805 (2006); N. S. Williams and A. N. Jordan, Phys. Rev. Lett. **100**, 026804 (2008).

[27] A. N. Jordan and A. N. Korotkov, Phys. Rev. B **74**, 085307 (2006).

[28] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, New York, 2001).

[29] N. S. Williams and A. N. Jordan, Phys. Rev. A **78**, 062322 (2008).

[30] It is easy to generalize Eqs. (53)–(56) to include the detector non-ideality and cross-correlation between the output and back-action noises (as in Ref. [24]). Non-ideality leads to the extra dephasing term $-\gamma_d\sigma_{12}$ in Eq. (56) for $\dot{\sigma}_{12}$, while the cross-correlation brings the term $iK[I(t)-I_0]\sigma_{12}$ into the same equation. The formulation of Eqs. (53)–(56) corresponds to the approach developed in Ref. [31].

[31] H. Wei and Yu. V. Nazarov, Phys. Rev. B **78**, 045308 (2008).

[32] J. M. Martinis, S. Nam, J. Aumentado, and C. Urbina, Phys. Rev. Lett. **89**, 117901 (2002); K. B. Cooper, M. Steffen, R. McDermott, R. W. Simmonds, S. Oh, D. A. Hite, D. P. Pappas, and J.M. Martinis, Phys. Rev. Lett. **93**, 180401 (2004).

[33] N. Katz, M. Ansmann, Radoslaw C. Bialczak, E. Lucero, R. McDermott, M. Neeley, M. Steffen, E. M. Weig, A. N. Cleland, J. M. Martinis, and A. N. Korotkov, Science **312**, 1498 (2006)

[34] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[35] L. P. Pryadko and A. N. Korotkov, Phys. Rev. B **76**, 100503(R) (2007).

[36] Strictly speaking, the QPT language may not be applicable to the uncollapsing experiment, because it requires linearity of the quantum operation, while after selection of particular realizations and state normalization the quantum operation is not necessarily linear. However, the linearity is preserved when the selection probability does not depend on the initial state, which is exactly the case for a perfect uncollapsing – see Eq. (18). It is also possible to show that even in the imperfect non-linear case the uncollapsing fidelity defined via the "naive" QPT language practically coincides with the rigorous definition via the average state fidelity.[37]

[37] K. Keane and A. N. Korotkov, in preparation.

[38] A. N. Korotkov, Phys. Rev A **65**, 052304 (2002);

[39] A. N. Jordan, B. Trauzettel, and G. Burkard, Phys. Rev. B **76**, 155324 (2007).

[40] F. H. L. Koppens, C. Buizert, K. J. Tielrooij, I. T. Vink, K. C. Nowack, T. Meunier, L. P. Kouwenhoven and L. M. K. Vandersypen, Nature **442**, 766 (2006).

[41] J. Elzerman, R. Hanson, L. H. Willems van Beveren, B. Witkamp, L. M. K. Vandersypen, L. P. Kouwenhoven, Nature **430**, 431 (2004).

[42] R. Hanson, L. H. Willems van Beveren, I. T. Vink, J. M. Elzerman, W. J. M. Naber, F. H. L. Koppens, L. P. Kouwenhoven, and L. M. K. Vandersypen, Phys. Rev. Lett. **94**, 196802 (2005).

[43] A. C. Johnson, J. R. Petta, J. M. Taylor, A. Yacoby, M. D. Lukin, C. M. Marcus, M. P. Hanson, A. C. Gossard, Nature **435**, 925 (2005); F. H. L. Koppens, J. A. Folk, J. M. Elzerman, R. Hanson, L. H. Willems van Beveren, I. T. Vink, H. P. Tranitz, W. Wegscheider, L. P. Kouwenhoven, and L. M. K. Vandersypen, Science **309**, 1346 (2005).

[44] K. Ono, D. G. Austing, Y. Tokura, and S. Tarucha, Science **297**, 1313 (2002); A. C. Johnson, J. R. Petta, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Phys. Rev. B **72**, 165308 (2005).

[45] Y.-S. Kim, Y.-W. Cho, Y.-S. Ra, and Y.-H. Kim, arXiv:0903.3077.